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Charles Clifford Conroy, A.M., Sc.D., Professor of Astronomy, Los Angeles College, California, U.S.A., P.O. box 826, Los Angeles;

Dudley Eglinton, Walmsley Street, Kangaroo Point, Brisbane, Queensland, Australia;

George James Gibbs, M.Inst.M.E., 7 Lune Street, Preston, and Brownedge, Bamber Bridge, *viâ* Preston, Lancashire;

James Martin Hatfield, 1 Palewell Park, East Sheen, S.W.;

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Commander Robert Cumming McFee, R.N.R., 37 St. Vincent Crescent, Glasgow;

Frederick Poate, Surveyor-General of New South Wales, Sydney, Australia;

William Herbert Steavenson, Ariel Lodge, Cheltenham; and
Harold Thomson, 37 Larkspur Terrace, Jesmond, Newcastle-upon-Tyne,

were balloted for and duly elected Fellows of the Society.

The following candidates were proposed for election as Fellows of the Society, the names of the proposers from personal knowledge being appended:—

Rev. W. Phillips, M.A., Presbyterian Minister, Normanhurst, Clement Avenue, Llandudno, North Wales (proposed by Robert Owen); and
John Miles Rogers Todd, Barrister-at-Law, 4 Temple Gardens, Temple, E.C., and The Lodge, Hextable, Kent (proposed by Robert S. Ball).

Eighty-two presents were announced as having been received since the last meeting, including, amongst others:—

A. E. H. Love, Some Principles of Geodynamics, Adams Prize Essay, presented by the Author.
Twenty Charts of the Astrographic Chart of the Heavens, presented by the San Fernando Observatory.

Hypothetical Parallaxes of the brighter stars of type A.

By H. C. Plummer, M.A.

1. Some months ago Dr. Campbell very kindly sent me in MS. a list of the radial velocities of a number of the brighter stars of types A and B observed at Mount Hamilton or Santiago under his direction, or collected from other sources. Dr. Campbell has himself made a thorough investigation of the motions of the stars of type B (*L.O.B.*, 195) and a more general study of the stars of all types (*L.O.B.*, 196). From the latter memoir a part of Table VI. may be reproduced here:—

Residual Velocities in terms of Galactic Latitudes.

	±90°–±60°		±60°–±30°		±30°–0°	
Class.	No.	km.	No.	km.	No.	km.
B	7	5·4	27	5·6	191	7·1
A	18	5·6	61	9·2	98	13·0
A to M	108	12·4	298	14·8	540	15·7

The most noticeable feature is shown by the class A stars: "the evidence afforded by the class A stars is very strong, to the effect that the radial velocities of the class A stars near the Milky Way are appreciably greater than the radial velocities of the stars in high latitudes." This suggests that the stars of this type have a marked tendency to move parallel to the plane of the Milky Way. In fact, if we were to suppose the stars scattered uniformly over the sphere and to be moving strictly parallel to this plane, it is easy to find that we should have the numbers—

$\pm 90° - \pm 60°$		$\pm 60° - \pm 30°$		$\pm 30° - 0°$	
No.	km.	No.	km.	No.	km.
26	4·6	72	9·7	98	13·0

where the numbers for the low latitudes are left unaltered. The resemblance to the figures actually found seems striking. When all the stars of types A–M are taken together, on the other hand, this feature is no longer apparent. It therefore seemed worth while to make a tentative assumption that the individual stars of type A are actually moving parallel to the plane of the Milky Way, and to examine the results which follow from this hypothesis.

2. The method which gives the results is simple. We know three rectangular components of the velocity of each star relative to the Sun in the form V, the radial velocity, $\mu_\alpha \cos\delta$ and μ_δ the proper motions. Let ϖ be the parallax. The units in which these are expressed are

Linear velocity	1 km./1 sec.
Parallax	1″
P.M.	1″/1 year.

We calculate the components B_1 and B_2 of μ_δ and $\mu_\alpha \cos\delta$ respectively normal to the plane of the Milky Way, and of C_1 the component of V in the opposite direction. Then we have

$$C + C_1 = [\cdot 6758](B_1 + B_2)/\varpi$$

where C is the assumed velocity of the Sun in the direction in which C_1 is measured, and the numerical factor (expressed as a logarithm) is inserted in order to express the right-hand side in the proper unit. The pole of the Milky Way has been assumed at

R.A. 192° Decl. + 27°

(Newcomb), and the Sun's velocity has been taken as 16·8 km. in the direction

R.A. 270° Decl. + 30°

(Campbell). This gives a component + 6·5 km. for the Sun normal to the Milky Way, and the simple assumption on which this work has been based may be expressed shortly by saying that relative to the Sun all the stars of type A are supposed to move from the pole of the Milky Way with this velocity, while their

velocities parallel to this plane may be any whatever. The above equation then determines the parallax of each individual star. When the parallax has been thus found, we have the three components of linear velocity expressed in linear measure, and hence the means of finding the corresponding components relative to three fixed axes through the Sun. The axes chosen are in the direction of 0 hours and 6 hours in the plane of the equator and towards the North Pole. These components are denoted by u, v and w respectively. The means of u, v, w for 121 stars are

$$U = +2\cdot00, \qquad V = +12\cdot01, \qquad W = -5\cdot60$$

giving

$$R = 13\cdot40 \text{ km.}, \qquad \alpha = 80^\circ\ 32' \qquad \delta = -24^\circ\ 41'$$

as compared with the initial assumption as to the Sun's motion reversed,

$$R = 16\cdot8 \text{ km.}, \qquad \alpha = 90^\circ \qquad \delta = -30^\circ.$$

The agreement is not perfect, but it is good enough to suggest that the assumed component, 6·5 km., is not very far from the truth if there is any justification for treating this as an approximate constant for the majority of the stars.

3. It is difficult to define with any precision the accuracy of the results of a pure hypothesis like the present. Relative parallaxes have been determined by direct observation in very few cases, and even in these the results are generally inconsistent and untrustworthy. But a certain number of failures present themselves definitely. The state of things may be expressed thus:—

Negative parallaxes	38
Excessive velocities	8
Excessive parallaxes	2
Reasonable cases	124
Total	172 stars

Velocities have been considered "excessive" which exceed the limit 60 km. a second. It is more difficult to assign a limit to the parallaxes. The number exceeding 0″·1 is 15, and these may be tabulated:—

Star.	Mag.	ϖ.	Star.	Mag.	ϖ.
		″			″
ζ Hydri	4·9	0·263	ω Herculis	4·5	0·165
ω Tauri	4·8	·205	α Lyræ	0·1	·263
Sirius	−1·6	·596	α Aquilæ	0·9	·113
Castor	2·8–2·0	·123	ϵ Cephei	4·2	·158
ι Ursæ Maj.	3·1	·157	β Piscis Austr.	4·4	·277
α Cancri	4·3	(·853)	α ,,	1·3	·488
β Leonis	2·2	·394	λ Piscium	4·6	·265
45 Bootis	5·0	(·797)			

Of these the two numbers bracketed are obviously excessive, and these are the two which have been definitely excluded. Of the rest, so far as can be judged, the parallaxes of Sirius, β Leonis, and Fomalhaut are certainly too large, while those of Castor, α Lyræ, and Altair are very likely near the truth. It is very improbable that a star of this type fainter than 4·0 mag. has a parallax $>0''{\cdot}2$, and no importance can be attached to the results which contradict this rule. But they have not been absolutely rejected The number of stars with parallaxes $<0''{\cdot}1$ is 111, and for these, with few exceptions, no test by comparison with other results is possible. But this proportion of small positive parallaxes, combined with reasonable linear velocities, is itself a good confirmation of the hypothesis that the majority of the stars of type A have a strong tendency to move parallel to the plane of the Milky Way.

4. When the components of the linear velocity of each star have been obtained, it is natural to make an attempt to group together those which seem to share a common motion. This can only be done tentatively. Two or three stars are noticed which have similar components. Others are added after an inspection of the whole list. Then a second group is formed in the same way, and a third, and so on. The approximate means for each group now become apparent, and it is possible to assign many of the stars remaining to one group or another. In all ten groups were formed in this way. Those stars added at a late stage often gave rather large residuals, but it seemed better to spoil the agreement among the members of a group rather than to multiply the number of the groups, since the total number of stars available is not large. Finally, a dozen stars remained over which could not be forced into any one of the groups. The four largest groups are given here in tabular form :—

[Table.

(i)

22 Stars. Star.	ϖ.	U = +3·7. u − U.	V = +8·0. v − V.	W = −4·0. w − W.
	"			
L 855	0·035	−1·1 km.	−4·4 km.	−3·7 km.
ζ Hydri	·263	−1·1	−4·5	−4·0
δ Monoc.	·010	+1·0	+3·5	+3·5
Castor	·123	+1·1	−2·1	+1·2
15 Ursæ Maj.	·095	+1·0	−3·6	+0·4
u Velorum	·023	+3·7	−5·1	+5·1
β Leonis	·394	+1·1	−2·4	+1·3
3 Corvi	·036	+1·7	+1·0	+3·6
ν^2 Bootis	·021	−3·5	+0·5	−6·4
ψ Scorpii	·034	+0·7	+1·3	+1·9
d ,,	·075	+0·8	+5·9	+3·8
19 ,,	·018	−2·3	+3·2	−3·0
ω Herculis	·165	+0·5	−1·9	+0·2
60 ,,	·056	+1·8	−4·7	+1·5
ρ ,,	·092	−3·6	+4·0	−5·2
α Lyræ	·263	−2·2	+7·1	−1·3
L 8245	·008	−2·2	+3·3	−2·8
ν Capric.	·008	−2·6	−1·6	−5·7
γ Gruis	·034	+0·8	+7·4	+4·6
ε Cephei	·158	+0·9	+4·8	+3·9
β Piscis Austr.	·277	+1·9	−9·4	−2·5
α ,,	·488	+1·2	−6·2	−0·3

(ii)

19 Stars. Star.	ϖ.	U = +2·0. u − U.	V = +37·5. v − V.	W = +4·8. w − W.
	"			
ν Fornacis	0·001	−5·4 km.	+6·3 km.	−7·8 km.
μ Ceti	·040	−2·9	+4·9	−3·5
64 Tauri	·036	+0·1	+2·4	+1·1
κ' ,,	·030	+0·6	+5·4	+3·3
68 ,,	·043	+2·0	−1·2	+3·4
ι ,,	·049	+2·5	+2·6	+6·0
Lal. 10063	·009	−3·5	−4·9	−8·9
L 3081	·004	−4·2	−4·6	−9·9
40 Hydræ	·004	−0·5	+14·6	+5·1
λ Ursæ Maj.	·022	−1·4	+2·4	−1·7
Brisb. 3689	·010	−1·4	−7·9	−6·1
γ Centauri	·025	−0·6	+0·9	−0·7
υ Ophiuchi	·043	+1·7	−6·8	+0·4
π Pavonis	·031	+0·6	−1·8	+0·5
L 7976	·001	+6·7	−9·0	+9·2
α Aquilæ	·113	+2·0	+4·7	+5·8
θ ,,	·008	−0·7	−2·7	−2·4
θ Indi	·020	−1·1	−?·1	−3·0
ρ Pegasi	·009	+0·2	+2·8	+1·5

(iii) 16 Stars. U = −3·6. V = +12·5. W = −17·2.

Star.	ϖ.	u − U.	v − V.	w − W.
	″			
α Androm.	0·052	−1·8 km.	+0·2 km.	−2·5 km.
β^2 Tucanæ	·060	+3·3	−5·5	+5·2
γ Arietis	·037	+1·4	−2·2	+2·9
β ,,	·035	+0·6	+0·5	+2·5
α Doradus	·019	−3·6	+5·3	−3·6
L 3197	·003	+0·2	−4·2	−0·3
e Velorum	·011	+0·1	+1·4	+1·8
n ,,	·022	−0·2	+1·9	−1·5
a ,,	·016	−2·0	+1·0	−2·4
ρ Hydræ	·010	−0·7	+3·4	+1·0
32 ,,	·009	+2·7	+0·7	+6·7
η Leonis	·002	−1·5	−6·4	−4·5
θ Virginis	·015	+1·4	−0·1	+3·8
L 5950	·021	+1·2	−1·6	+2·7
β Circini	·056	+1·1	−2·8	+2·1
κ Piscium	·032	+1·3	+0·7	+3·9

(iv) 13 Stars. U = +6·7. V = −6·2. W = −4·0.

Star.	ϖ.	u − U.	v − V.	w − W.
	″			
32 Persei	0·029	+1·5 km.	+1·4 km.	+3·5 km.
Pi 57	·015	+3·6	+2·2	+7·7
β Aurigæ	·055	−3·1	−7·5	−9·2
Sirius	·596	−0·7	−3·7	−2·8
L 2690	·045	−2·0	−1·1	−4·3
α Volantis	·065	−2·6	+3·6	−3·6
n Centauri	·031	+1·3	−2·5	+1·5
π′ Bootis	·007	+2·6	−1·0	+4·7
ν Serpentis	·017	+2·1	−2·6	+3·0
L 8207	·019	−0·2	+2·5	+0·7
α Cygni	·0005	−4·3	+3·3	−6·9
o Pegasi	·035	+1·4	+2·2	+3·6
λ Piscium	·265	+0·2	+3·0	+1·6

5. These four groups account for 70 stars. It is not claimed that all these stars have been rightly placed. Indeed there are reasons for suspecting the contrary in some cases, for example the members of Hertzsprung's enlarged Ursa Major group; and some of the residuals are certainly large. But the classification has a certain suggestive value, which may prove to be considerable, and

is in accordance with Kapteyn's conclusion that the stars of early type tend to form streams. The resultant velocities relative to the Sun and the convergents of the four groups are as follows:—

Group.	No.	R.	A.	D.
i	22	9·7 km./sec.	65°·2	−24°·4
ii	19	37·9	87·0	+ 7·3
iii	16	21·6	106·1	−52·9
iv	13	10·0	317·2	−23·6

Of these, group ii is perhaps the most interesting. The corresponding figures for the Taurus stream discovered by Professor Boss are—

$$R = 45{\cdot}6, \qquad A = 91^{\circ}{\cdot}6, \qquad D = +6^{\circ}{\cdot}9.$$

The convergent is in good agreement, and if the velocity is greater it may be noted that Professor Boss used the results for three stars of late type observed at Bonn. Indeed the stars 64, κ', and 68 Tauri are common to the two lists. It would therefore appear that the Taurus stream is no mere localised cluster, but contains members which are scattered over the whole sky. If this result should be confirmed by later work it will be a most interesting fact. At present it must be insisted on that this conclusion, like all others in this paper, is the outcome of the application of an hypothesis which we are aware is only approximately true.

6. The proper motions employed in this work have been taken from Boss's *Preliminary General Catalogue*. My thanks are due to Dr. Campbell for communicating to me his list of radial velocities, and to Professor Turner for facilities in carrying out the necessary calculations. It is hoped that a fuller account of the results will be published later.

The Constitution of the Ring Nebula in Lyra (*N.G.C.* 6720).
By J. W. Nicholson, M.A., D.Sc.

In the paper on "The Spectrum of Nebulium," published in the *Monthly Notices* of November last, it was shown that this hypothetical substance could give an account of all the unidentified nebular lines tabulated by Wright* with two exceptions. The tabulated values were the results of Wright's investigation of the nebula in Orion, and of G.C. 4390, G.C. 4964, N.G.C. 7027. The two lines of which an account was not given are of wavelengths 4685·7 and 3729·0.

By the kindness of Dr. Max Wolf, of Heidelberg, the writer has been furnished with an account of some experimental observations of the Ring nebula in Lyra, which appear to throw a

* *Lick Observatory Bulletin*, No. 19; *Ast. J.*, xvi. (1902).

valuable light on the nature of these two lines, and on the cause of their absence from the calculated nebulium spectrum. Dr. Wolf's observations,* which were communicated to the Heidelberg Academy on the same day as the writer's paper to the Society, show that, in the Ring nebula, the maximum emission for the various wave-lengths takes place in different strata, so that the substances giving rise to them have a definite spatial arrangement, perhaps in order of density. This is undoubtedly proved by the beautiful experiments which Dr. Wolf describes.

Two of the results are of special interest for the present purpose. Dr. Wolf has found that the central dark space of the nebula, after a long exposure, affects the plate, and that the radiation of wave-length 4686 arises solely from this space, and not from the bright ring.

The maximum emission for the wave-length 3729 occurs towards the outer edge of the bright ring, although extending more faintly throughout the ring itself.

The maxima for other wave-lengths occur inside the bright ring, and it is not possible, when account of the order of accuracy is taken, to say definitely whether they differ very appreciably in position. But with regard to the two wave-lengths now in question, the results are decisive, and appear to indicate that they are not due to the substance responsible for the other nebular lines, and, moreover, that they are themselves due to two different substances. Dr. Wolf suggests that there may be a considerable difference in the atomic weights of these substances.

When this evidence is considered together with the writer's paper on the Spectrum of Nebulium, it is evident that nebulium can account for all the connected unknown lines of Wright's table without exception, and the view that they are actually caused by it is considerably strengthened. As to the nature of the substances which lie in the dark space and perhaps beyond the bright ring of the Ring nebula, it is very difficult to conjecture, for each is not sufficiently luminous to show more than its strongest line. Apparently, however, the interior substance may be denser than either hydrogen or nebulium, and it may be a terrestrial element formed by combination from the more primary forms. Mr. C. P. Butler, in the discussion on the writer's paper, also stated that he had noted peculiarities in this line 4686, and that there was reason for believing that it could be produced in the laboratory.

* *Sitz. der Heidel. Akad.*, Jahrgang 1911, Abhand. 27.

1912 *January* 7.

Personality and Bisection Error of some Greenwich Transit Circle Observers. By W. G. Thackeray.

(*Communicated by the Astronomer Royal.*)

The immediate object of this paper was to obtain some determination of the magnitude equation of the present regular observers with the Transit Circle, and as far as R.A. is concerned the resulting magnitude equation is obtained by a comparison between the Greenwich Transit Circle results for 1906–1910 and the star places of Boss's and Küstner's Catalogues for 1900. The North Polar distances show no magnitude equation, but systematic differences between the different observers, which led up to a more detailed examination of the Greenwich Transit Circle reductions.

The results seem to suggest that N.P.D. observations with a Transit Circle require some further correction not hitherto applied.

The Greenwich observations for magnitude equation have been divided into two groups, clock stars and zone stars, the latter being the reference stars of the Oxford astrographic zone N.P.D. 58° to 66°. The clock star observations have been compared with Boss's standard star places for 1910; the zone stars with Küstner's Catalogue of 10,663 stars for 1900. There are about 1000 of these latter common to the two catalogues, and they have been reduced to 1910, with the use of proper motions when known.

The two groups have been made comparable by applying to the results Küstner–Greenwich, for the zone stars, the corrections $+0^{s}\cdot010$, $+0''\cdot31$ derived from the tables given in Boss's Catalogue of 6188 stars for 1900, Appendix III.

The Greenwich Right Ascensions depend on the standard R.A.'s of the Second Greenwich Nine Year Catalogue for 1900·0, and the North Polar distances are reduced to a uniform system with Pulkowa refractions, colatitude $21''\cdot80$ and an adopted value of R – D of $0''\cdot60$ sin Z.D., and further corrected where necessary for the division errors adopted in 1898 and for latitude variation.

The following table exhibits the differences in R.A. and N.P.D. for each observer, with the number of observations subscribed :—

[Table.

Table I.

Boss–Greenwich Observers 1906–10.

Limits of Magnitude.	Right Ascensions. W.	J. S.	E.	R. C.	B. E.	North Polar Distance. W.	J. S.	E.	R. C.	B. E.
	(unit ^{s}·001.)									
Clock Stars.										
0·0 – 0·9	−15	+9	+2	+3	+6	+·42	−·49	+·21	+·10	+·74
	35	19	18	43	12	39	19	26	48	8
1·0 1·9	+5	+19	+3	+15	+10	·61	+·13	·38	·19	·16
	70	28	36	58	17	56	28	42	51	24
2·0 2·9	−5	−3	+8	−7	+6	·34	·13	·51	·11	·26
	181	105	122	158	50	132	97	111	152	66
3·0 3·9	−1	−4	−3	−7	−7	·36	·07	·40	·27	·26
	214	125	168	221	68	142	112	142	188	87
4·0 4·9	+4	−1	−9	+2	−12	·68	·28	·42	·27	·28
	180	128	172	203	80	121	98	147	159	113
5·0 5·9	−6	−13	−9	−8	−4	·35	·06	·58	·09	·48
	75	45	58	75	46	46	52	50	60	54
6·0 6·9	+2	−16	+2	−2	+5	·89	·07	·61	·31	·47
	13	18	16	28	10	11	18	18	19	15
						Mean +·46	+·11	+·44	+·22	+·31
						547	424	536	617	367
Zone Stars.										
5·0 – 5·9	−32	+19	−27	+9	+2	+·44	+·31	+·66	+·61	+·40
	14	22	21	8	22	14	22	21	8	17
6·0 6·9	−9	+1	−9	+5	−12	·42	·36	·32	·11	·03
	24	22	19	16	13	24	22	19	16	13
7·0 7·9	−23	−10	−32	+8	+3	·67	·16	·39	·31	·44
	43	24	45	34	46	43	24	45	34	46
8·0 8·4	−56	−5	−20	−17	+7	·50	·14	·31	·34	·50
	83	32	51	61	58	83	32	51	61	58
8·5 8·9	−56	−30	−37	−10	−31	·65	·50	·51	·36	·23
	154	32	55	87	54	154	32	55	87	54
9·0 ...	−87	−58	−65	−27	−5	·67	·30	·47	·47	·41
	122	13	14	62	15	122	13	14	62	15
						Mean +·61	+·30	+·43	+·37	+·37
						440	145	205	268	203

The above table shows that in R.A. the various observers have little or no personality down to $6^{m}\cdot5$ at any rate but after that largely increasing variations, in N.P.D. no magnitude personality but constant systematic differences of a nature which seems to indicate bisection error or some such discordance.

Examination was first made of the N.P.D.'s of the nine azimuth stars, and the observations above and below pole grouped for each observer. This showed at once systematic differences between the different observers, and suggested that the R – D discordances if arranged for different observers would also show systematic differences. The R – D discordances were then arranged in groups of years chosen so as to meet apparent changes

in the value of the R – D, as well as change of observers, and new values determined for each observer. These results showed not only considerable variations in the sin Z D term, but also in the constant term. This implied that what is commonly called in the Greenwich introductions correction to "Wire R" would vary according to the observer. This was found to be the case. The correction to wire R is really the difference of the horizontal point of the Transit Circle as determined by reflexion stars observed at equal distances north and south of the zenith, and that determined by observations of the nadir, and may, of course, be corrections to the one or the other. Assuming that bisection error means any systematic discordance which affects the use of the zenith distance micrometer by the observer, then if this bisection error is constant or varies with zenith distance equally for an R and D observation, such error will go out in the adopted zenith point from reflexion observations, but will remain in the resulting R and D zenith distances, and the discussion of the R – D discordances would determine the necessary corrections. If, however, the bisection error of the R and D observations is variable, or the bisection error varies on different sides of the zenith, then these variations will come into the adopted zenith points as well as the concluded zenith distances. The variations of the value of the sin Z D term shown in Table II. would naturally suggest a bisection error varying with the zenith distance for different observers.

Hitherto it has been assumed that the personal errors of an observer may be regarded as accidental, and that in the main the reduction to the general mean observer will fairly represent the actual facts, and probably this principle in the long run may be more or less right, but it is evident from the personal differences shown to exist in the following tables that with changes of observers the value of the R – D discordance may still remain fairly constant according as the personalities of the observers are much the same or balance one another in the mean, or, on the other hand, it may abruptly change with a change of observers.

With reference to Table III. it will be noticed that the alterations to the object-glass were in 1891 coincident with a change, and in 1906 with no change in the observers.

Both Tables II. and III. seem to show that the quantities determined are largely dominated by personality, and are not a direct measure of instrumental error.

TABLE II.

Values of R – D. Formula a + b *sin* Z D *for each Observer for different Groups of Years.*

Group of Years.	Value of *a* term.										Value of sin Z D term.									
	A. D.	T.	L.	H.	A. C.	B.	W.	J. S.	E.	R. C.	A. D.	T.	L.	H.	A. C.	B.	W.	J. S.	E.	R. C.
1887–91	″·00	″–·01	″–·02	″+·02	″ ...	″ ...	″ ...	″ ..	″ ...	″ ...	″+·35	″+·70	″+·71	″+·87	″ ...	″ ...	″ ...	″ ...	″ ...	″ ...
*1892–96	...	...	...	+·04	+·09	+·11	...	...	...	...	...	...	...	+·74	+·38	+·44	...	...	...	...
1897–01	...	...	...	+·03	+·12	+·08	...	...	...	...	...	...	...	+·45	–·07	+·16	...	...	...	...
1902–05	...	...	...	...	...	+·09	+·07	+·04	...	...	...	...	...	...	...	+·28	+·36	+·14	...	...
*1906–10	...	...	...	...	...	...	+·07	...	+·05	+·06	...	...	...	...	...	..	+·53	...	+·83	+·40

* The object-glass was re-polished in 1891 October and 1906 February and April.

TABLE III.

Corrections to Wire R for different Observers.

Years.	A. D.	T.	L.	H.	A. C.	B.	W.	J. S.	E.	R. C.	Adopted Mean.
	″	″	″	″	″	″	″	″	″	″	″
1887	+·06	−·21	+·15	−·16	...	...	...	...	...	...	+·01
1888	−·24	−·15	+·07	−·14	. .	...	...	...	...	...	−·12
1889	+·03	+·12	+·05	+·05	...	...	...	...	...	...	+·07
1890	−·08	+·23	+·25	+·08	...	...	...	...	...	...	+·08
1891*	+·36	+·04	+·74	+·09	...	...	...	...	...	...	+·07
1892	...	...	+·22	−·11	−·53	−·38	...	...	...	...	−·25
1893	...	...	...	−·28	−·35	−·51	...	...	...	...	−·34
1894	...	...	...	−·27	−·33	−·45	...	...	...	...	−·27
1895	...	...	...	+·02	−·55	−·42	...	...	...	...	−·31
1896	...	. .	...	+·05	−·61	−·44	...	...	...	...	−·34
1897	...	...	...	·00	−·46	−·43	...	...	...	...	−·27
1898	...	...	...	−·13	−·57	−·45	...	...	...	...	−·36
1899	...	...	...	−·45	−·54	−·57	...	...	...	...	−·41
1900	...	...	...	−·29	−·60	−·46	...	...	...	...	−·39
1901	...	...	...	−·21	−·49	−·40	...	...	...	...	−·29
1902	...	...	...	...	...	−·49	−·43	+·06	...	...	−·20
1903	...	...	...	...	...	−·43	−·30	−·14	...	...	−·23
1904	...	...	...	...	...	−·52	−·35	−·46	−·51	...	−·41
1905	...	...	...	...	...	−·21	−·33	−·10	−·18	...	−·22
1906*	...	...	...	...	...	...	−·46	−·28	−·18	−·37	−·40
1907	...	...	...	...	...	...	−·23	−·22	−·11	−·45	−·24
1908	...	...	...	...	...	...	−·48	...	−·23	−·19	−·33
1909	...	...	...	...	...	...	−·34	...	−·22	−·08	−·16
1910	...	...	...	...	...	...	−·35	...	−·11	−·40	−·20

The zenith point at Greenwich is derived from observations of the reflected image of the horizontal wire and direct and reflected images of stars at the same transit. These results are divided into three groups, north stars, south stars and nadirs; a weight of two is given to each group of stars and a weight of one to the mean of the nadirs; thus any discordance between the zenith point as depending on nadir observations alone would enter into the combined zenith point with a weight of one-fifth.

For some years it has been the practice to keep each observer's zenith point separate, and a correction of $-0''\cdot25$, derived from a comparison between the nadir and reflexion observations, has

* The object-glass was repolished in 1891 October, and in 1906 February and April.

been applied to the observations of the nadir on those days where no reflexion observations of stars have been obtained. If reference is made to the introduction of the Greenwich Second Nine Year Catalogue for 1900·0, p. 18, particulars will be found of the changes of the flexure term derived from the discussion of the R − D discordances, and the consequent adoption of the value of 0″·60 sin Z D for the years 1897–1905, a value which has been used in the Greenwich reductions up to the present time.

In determining the zenith point from observations of the nadir, the value of the flexure is obtained from observations on the collimators. The value of the flexure of the Greenwich Transit Circle so determined has always been practically 0″·00.

The observations of the regular Greenwich Transit Circle observers for the years 1897–1910 have been divided into three groups of years 1897–1901, 1902–1905, and 1906–1910, representing three different groups of observers. South of the zenith all clock stars between N.P.D. 60°–110° have been compared with Boss's standard places for 1910, and tabulated for each observer for every 10° of N.P.D.

North of the zenith, the observations above and below pole of the nine azimuth stars were selected as affording the best material for obtaining a check on the observer's bisection error on this side of the zenith, and the results for each observer have been grouped for the same periods and are given as corrections to the adopted colatitude 21″·80.

The observations are reduced on the following four suppositions:—

(1) Ordinary zenith point without R − D.
(2) „ „ with adopted R − D (60 sin Z D).
(3) „ „ with observer's own R − D (Table II.).
(4) Zenith point from nadirs only flexure 0″·00.

The results (4) are obtained by applying to the results (1) $\frac{4}{5}$ of the quantities given as corrections to wire R (Table III.) and using the flexure derived from observations on the collimators.

[Table.

Table IV.

Boss minus Greenwich Observer arranged in Periods.

Limits of N.P.D.	1897–1901. H.	A. C.	B.	1902–1905. B.	W.	J. S.	1906–1910. W.	J. S.	E.	R. C.	Mean.
				(1) Without R – D.							
° °	″	″	″	″	″	″	″	″	″	″	″
60–70	+·52	+·46	+·48	+·69	+·86	+·37	+·66	+·46	+·63	+·47	+·56
70–80	·59	·55	65	·65	·70	·33	·76	·46	·79	·59	·61
80–90	·51	·45	·50	·50	·65	·24	·77	·51	·84	·64	·56
90–100	·87	·60	·77	·90	·91	·62	1·15	·75	1·05	·90	·85
100–110	+·96	+·87	+·86	+·98	+·80	+·61	+1·27	+·87	+1·23	+·96	+·94
Corr. to Col.	−·28	−·02	−·26	−·15	−·26	−·13	−·20	−·36	−·27	−·37	−·22
				(2) Adopted R – D.							
60–70	+·17	+·11	+·13	+·36	+·53	+·04	+·35	+·15	+·32	+·16	+·23
70–80	·16	+·12	+·22	·23	·28	−·09	·36	·06	·39	·19	·19
80–90	·00	−·06	−·01	·00	·15	−·26	·29	·03	·36	·16	·07
90–100	·30	+·03	+·20	·34	·35	+·06	·61	·21	·51	·36	·30
100–110	+·34	+·25	+·24	+·37	+·19	·00	+·68	+·28	+·64	+·37	+·34
Corr. to Col.	+·05	+·32	+·07	+·18	+·07	+·20	+·14	−·02	+·06	−·03	+·11
				(3) Observer's own R – D.							
60–70	+·27	+·37	+·32	+·47	+·62	+·26	+·35		+·21	+·22	+·34
70–80	·29	·47	·47	·41	·41	·21	·37		·24	·28	·35
80–90	·16	·38	·30	·21	·32	·10	·31		·19	·28	·26
90–100	·46	·54	·57	·57	·55	·46	·63		·31	·50	·51
100–110	+·52	+·82	+·63	+·63	+·40	+·44	+·71		+·42	+·53	+·57
Corr. to Col.	−·03	−·10	−·24	−·07	−·11	−·08	+·05		+·17	−·18	−·04
				(4) Nadir and Flexure 0″·00.							
60–70	+·69	+·88	+·84	+1·02	+1·12	+·49	+·96	+·66	+·77	+·71	+·86
70–80	·76	·97	1·01	·98	·96	·45	1·06	·66	·93	·83	·86
80–90	·68	·87	·86	·83	·91	·36	1·07	·71	·98	·88	·82
90–100	1·05	1·02	1·13	1·23	1·17	·74	1·45	·95	1·19	1·14	1·11
100–110	+1·13	+1·29	+1·22	+1·31	+1·07	+·73	+1·57	+1·07	+1·37	+1·20	+1·30
Corr. to Col.	−·11	+·40	+·10	+·18	·00	−·01	+·10	−·16	−·13	−·13	+·03

The tabulated quantities depend on the average on 50 to 150 observations.

The personalities shown to exist by these tables are considerable and systematic, and indicate variation of the bisection error

on different sides of the zenith. No system of reduction will make the observations accordant on both sides of the zenith, and a judicious selection of observer and system will give a correction to Boss varying somewhere between 0″·1 and 1″·0. It certainly seems reasonable to assume that reflexion observations should give the better results, for two reasons, first, that the zenith point depends on observations all over the circle instead of from one point alone, and secondly, the R and D observations afford material for determining the flexure of the instrument as well as the bisection errors of the observers. From the variation in the values of the flexure derived from different observers, it seems clear that the observer's bisection errors vary with zenith distance, and this partly explains why the flexure derived from the R−D observations should not agree with that derived from the collimators. An explanation of these personalities may possibly be found to be due partly to the eye and partly to the position of the body of the observer at the place of observation. The observer with the Transit Circle takes up three different positions according to the zenith distance of the star, recumbent, sitting, and standing; in addition to this, when making observations on the south of the zenith, he has to raise his arm above his head to move the telescope-micrometer, whereas on the north side he has merely to raise his hand, as the micrometer is just under his chin.

Observations of Occultations of Stars by the Moon, made at the Royal Observatory, Greenwich, in the Year 1911.

(*Communicated by the Astronomer Royal.*)

Day. 1911.	Phenomenon.	Mag.	Telescope.	G.M.T. h m s	Observer.
Mar. 9	Disapp. B.D. +26°1431	8·7	Astrographic	7 23 22·00	W. S.
9	,, B.D. +27°1308	8·1	,,	8 8 32·63	W. S.
9	,, Lalande 13593	8·0	,,	8 23 53·63	W. S.
9 (*a*)	,, ,,	,,	Merz	52·78	Wh.
9 (*a*)	,, B.D. +27°1315	9·4	Astrographic	9 49 18·70	W. S.
9	,, B.D. +26°1464	8·8	,,	10 51 34·05	W. S.
9	,, B.D. +26°1466	9·2	,,	10 53 55·16	W. S.
9	,, B.D. +26°1469	9·1	,,	11 3 38·78	W. S.
9	,, B.D. +26°1481	7·0	,,	11 54 39·46	W. S.
9	,, B.D. +26°1482	8·9	,,	12 9 25·15	W. S.
Apr. 3	,, W.B. (2) IV. 637–8	8·9	,,	7 28 25·99	Wh.
3	,, W.B. (2) IV. 802	8·9	,,	10 9 4·89	W. S.
5	,, W.B. (2) VI. 1224	7·1	,,	8 36 48·28	C. P.
8	,, W.B. (2) IX. 800	7·0	,,	7 43 19·82	C. P.
8	,, ,,	,,	,,	18·34	G.
Dec. 3	,, 29 Arietis	6·1	Sheepshanks	8 9 28·52	G. B.

(*a*) Faint.

The apertures of the telescopes used are as follows:—

Merz Refractor. . .	$12\frac{3}{4}$ inches.	Clock Arnold 1.
Guiding telescope of Astrographic Equatorial	10 ,,	,, Dent 2014.
Sheepshanks Equatorial .	$6\frac{3}{4}$,,	Time tapped on chronograph; $-0^{s}\cdot20$ applied.

The initials W. S., C. P., G., Wh., G. B., are those of Mr. Stevens, Mr. Pierce, Mr. Green, Mr. Wheate, and Mr. Bartle.

Note on certain Coefficients appearing in the Algebraical Development of the Perturbing Function. By R. T. A. Innes.

Near the top of page 196 of vol. lxx. (*Monthly Notices*, Dec. 1909), the formula

$$D^m b_{i,3} = a(D+1)^m[D^2+2D+1-i^2]b_{i,1}$$

is given.

To this should be added the complementary formula

$$a(D+1)^m b_{i,3} = D^m[D^2-i^2]b_{i,1}$$

which will afford valuable checks. A simple particular case is

$$ab_{0,3} = D^2 b_{0,1}$$

"Venus and Earth," *Ast. Papers of the American Eph.*, vol. v. p. 343.

$b_{0,3}$	a	$D^2b_{0,1}$
$7\cdot2279 \times 0\cdot723332$	$=$	$5\cdot2280$.

The extension to $b_{i,s}$ is sufficiently obvious.

The connection of the new formula with that on page 637 of vol. lxix. may be remarked.

On page 195 of vol. lxx., line 9 from bottom, $D+1$ is printed instead of $D+i$.

Johannesburg:
1911 *November* 24.

Sur la Longitude de l'Observatoire de Lille déterminée par Télégraphie sans fil. Par Robert Jonckheere.

Depuis le 7 Mars nous comparons presque journellement l'heure que nous recevons de la Tour Eiffel avec l'heure locale de notre Observatoire. Quoique ces signaux ne soient pas envoyés dans le but de déterminer rigoureusement les longitudes, nous les croyons assez exacts et la série d'observations assez longue pour faire connaitre le résultat de ces déterminations.

La moyenne pondérée de 252 signaux nous donne $3^m\ 23^s{\cdot}519$ pour la différence de longitude Est avec Paris. En adoptant $9^m\ 20^s{\cdot}95$ pour la différence Greenwich Paris, nous trouvons donc que notre Observatoire est de $12^m\ 44^s{\cdot}47$ à l'Est du méridien de Greenwich. L'erreur probable est de $\pm 0^s{\cdot}07$.

Il ne serait naturellement pas juste de comparer cette précision avec celle qui peut être obtenue par Télégraphie sans fil par la méthode des coïncidences; mais ces données sont intéressantes comme indiquant les résultats qui peuvent être obtenus d'une façon indépendante et par le seul envoi journalier des signaux horaires.

Pour montrer la précision pratique de ces signaux, voici quelques exemples isolés et pris au hasard :

Date.	Longitude trouvée.	Date.	Longitude trouvée.
	m s		m s
9 Octobre	3 23·40	6 Novembre	3 23·40
12 ,,	23·56	14 ,,	23·62
16 ,,	23·49	20 ,,	23·52
17 ,,	23·54	27 ,,	23·49
31 ,,	23·49	4 Décembre	23·52
3 Novembre	23·69	6 ,,	23·62

Il faut considérer que dans ces déterminations notre erreur est également comprise.

Afin de limiter autant que possible l'équation personnelle, des observations ont également été faites par M. J. Vanderdonck. Nous avons trouvé une différence assez constante de $0^s{\cdot}19$ entre nos observations qui étaient faites par la méthode de l'œil et de l'oreille. Cette différence étant supérieure à l'erreur probable des signaux, il semblerait que la précision habituelle des envois est suffisante lorsqu'il n'y a pas permutation entre les observateurs.

Nous ne pouvons qu'admirer la précision actuelle de l'envoi de l'heure par l'Observatoire de Paris. L'erreur probable est remarquablement faible étant donné la difficulté de maintenir constamment, pour ces envois, une pendule en temps absolu.

Observatoire de Lille :
1911 *Décembre* 20.

Nouvelles étoiles doubles. 8ème Série.

Par Robert Jonckheere.

Nous avons décidé de prendre une limite plus serrée ; au lieu de 5″, nous rejetons maintenant toutes les étoiles écartées de plus 3″. Le nombre de découvertes est forcément plus restreint, mais nous croyons notre équatorial suffisamment puissant pour nous spécialiser sur les couples les plus difficiles.

L'étoile J 677 est un compagnon plus serré à Σ 806.

J.	Catalogue.	Æ. 1920. h m s	Dec. ° ′	Date. 1911+	P. °	D. ″	Grandeurs.		n.	Obs.
668	+ 7° 4941	22 49 44	+ 7 57	0·929	336·3	1·37	9·4	10·0	1	J
				0·929	320·1	1·48	9·1	10·3	1	V
669	Anon.	22 53 10	+13 38	0·937	33·9	0·98	9·4	9·6	1	J
				0·937	37·2	0·88	9·4	9·4	1	V
670	Lpz. II. 1156	23 8 00	+ 8 26	0·929	188·7	1·63	8·8	10·4	1	J
				0·929	191·8	1·69	8·9	10·4	1	V
671	Anon.	1 48 30	+21 28	0·976	155·6	2·80	9·5	9·7	1	J
				0·976	152·7	2·72	9·5	9·7	1	V
672	+11° 370	2 35 57	+11 54	0·929	86·2	2·97	9·3	9·3	1	J
				0·929	85·1	3·28	9·3	9·3	1	V
673	+28° 486	2 59 10	+28 22	0·951	10·9	2·25	9·1	9·1	1	J
				0·951	10·0	1·88	...	...	1	V
674	+13° 855	5 14 19	+13 39	0·888	81·5	2·99	9·2	10·0	1	J
675	+21° 853	23 33	+21 50	0·954	188·7	2·06	9·2	9·7	1	J
				0·954	189·9	1·88	9·1	9·7	1	V
676	Lpz. II. 2271	31 11	+ 7 23	0·968	100·4	1·74	8·9	8·9	1	J
				0·968	98·0	1·30	9·0	9·0	1	V
677	Berl. A. 1725	46 26	+17 52	1·023	68·5	1·88	8·9	10·8	1	J AB
				1·023	69·3	2·16	9·0	10·8	1	V
	Σ 806			1·023	198·6	10·80	8·9	9·0	1	J AC
				1·023	198·7	10·97	9·0	9·1	1	V
678	Anon.	49 37	+14 26	1·017	188·9	2·25	10·0	11·0	1	J
				1·017	183·9	2·45	10·0	11·0	1	V
679	Anon.	49 56	+14 34	1·017	353·8	2·71	9·4	10·0	1	J
				1·017	357·5	2·56	9·4	9·8	1	V
680	Anon.	55 38	+16 9	1·017	166·9	2·98	9·7	11·5	1	J
				1·017	160·0	3·19	9·5	11·0	1	V
681	Anon.	5 57 7	+14 37	1·017	213·4	2·99	9·7	9·7	1	J
				1·017	212·9	3·25	9·8	10·0	1	V
682	−7° 1295	6 4 19	− 7 21	0·954	173·4	2·19	8·9	9·0	1	J
				0·954	172·4	1·87	8·9	9·4	1	V

J.	Catalogue.	Æ. 1920.	Dec.	Date.	P.	D.	Grandeurs.		n.	Obs.
		h m s	° ′	1911+	°	″				
683	Berl. A. 2011	6 9 56	+17 27	0·976	7·5	0·94	8·3	9·0	1	J
				0·976	5·5	1·12	8·5	9·3	1	V
684	+18° 1140	10 38	+18 15	1·017	69·0	2·50	9·0	9·9	1	J
				1·017	71·9	...	9·1	9·9	1	V
685	Anon.	12 21	− 6 32	0·954	175·4	1·93	9·4	9·4	1	J
				0·954	172·3	2·30	9·5	9·5	1	V
686	+22° 1258	12 36	+22 28	0·954	49·9	2·05	8·9	9·2	1	J
				0·954	45·6	1·89	8·9	9·5	1	V
687	Lpz. II. 2785	14 9	+ 7 47	0·924	171·5	2·07	9·0	9·0	1	J
				0·924	172·5	2·05	9·1	9·1	1	V
688	+21° 1224	19 51	+21 10	0·976	110·7	0·83	9·2	9·6	1	J
				0·976	108·9	1·06	9·2	9·7	1	V
689	Anon.	23 1	+ 8 28	0·987	143·9	1·80	9·3	13·0	1	J
690	+10° 1158	26 22	+10 7	0·924	357·4	1·80	8·8	11·0	1	J
				0·924	358·2	2·17	9·0	11·0	1	V
691	Alb. 2219	26 38	+ 4 43	1·023	203·7	0·99	8·8	9·7	1	J
				1·023	203·5	1·13	8·7	9·8	1	V
692	+8° 1385	28 31	+ 8 33	0·963	28·5	2·72	9·3	11·0	1	J
				0·963	27·3	2·58	9·4	11·0	1	V
693	+29° 1276	29 59	+29 14	0·987	271·1	1·35	9·1	9·3	1	J
				0·987	276·3	1·18	9·1	9·2	1	V
694	Anon.	33 45	+21 28	0·987	316·2	2·77	9·6	9·7	1	J
				0·987	319·0	2·76	9·6	9·6	1	V
695	+6° 1333	35 33	+ 6 2	1·023	155·0	1·85	9·0	11·0	1	J
696	−10° 1627	35 56	−10 30	0·924	163·4	1·30	9·1	9·2	1	J
				0·924	166·8	1·38	9·4	9·4	1	V
697	Lpz. I. 2451	39 6	+11 14	1·023	187·3	1·58	8·6	9·9	1	J
				1·023	186·9	1·91	8·8	10·0	1	V
698	Anon.	46 9	+ 5 47	0·954	234·6	2·70	9·1	10·0	1	J
				0·954	240·9	2·75	9·0	10·2	1	V
699	+12° 1330	48 5	+12 18	1·017	252·9	1·17	8·7	9·0	1	J
				1·017	256·4	1·26	8·9	9·1	1	V
700	Anon.	6 50 20	+10 20	0·954	104·7	2·19	9·5	9·5	1	J
				0·954	103·8	2·13	9·5	9·6	1	V
701	Anon.	7 3 3	+24 24	0·954	275·3	1·86	9·3	9·6	1	J
				0·954	276·8	1·77	9·5	9·6	1	V
702	+5° 1560	5 30	+ 5 34	1·023	267·5	2·92	8·9	9·7	1	J
				1·023	270·6	2·80	8·9	9·7	1	V
703	Anon.	7 6 4	+15 51	1·012	153·1	2·31	9·7	9·7	1	J
				1·012	149·3	2·51	9·6	9·6	1	V

J.	Catalogue.	Æ. 1920.	Dec.	Date.	P.	D.	Grandeurs.	n.	Obs.
		h m s	° ′	1911+	°	″			
704	Lpz. II. 3586	7 6 56	+ 5 30	1·023	193·8	2·72	9·1 10·0	1	J
				1·023	192·5	2·71	9·2 10·5	1	V
705	+4° 1640	11 19	+ 4 39	0·954	12·7	2·64	8·9 9·2	1	J
				0·954	12·6	2·57	8·8 9·0	1	V
706	+18° 1634	24 39	+18 27	0·968	183·0	2·99	8·9 10·0	1	J
				0·968	185·6	3·14	9·0 10·2	1	V
707	+11° 1672	7 42 16	+11 17	0·954	122·0	1·81	9·0 9·0	1	J
				0·954	120·1	2·03	9·1 9·1	1	V

Les étoiles suivantes ont été découvertes par M. J. Vanderdonck :

1	+26° 1293	6 31 31	+26 48	0·924	41·9	2·87	9·1 10·2	1	V
				0·924	40·8	2·98	9·1 10·0	1	J
2	Lpz. II. 11445	22 50 25	+ 7 58	0·929	126·8	2·95	9·0 9·0	1	V
				0·929	128·2	2·99	9·2 9·2	1	J

Notes.

679.—Une 10e grandeur à 188°·2, 9″·5.

680.—A côté d'une étoile de 9·3 B.D.

Les couples A 2301 et A 2303 récemment publiés, sont identiques à J 219 et J 585 respectivement.

Errata.—Dans les notes page 161, pour Hu 74, lire Hn 74. Notes page 165, pour 649, lire 666.

Observatoire de Lille,
le 10 Janvier 1912.

Micrometrical Measures of Double Stars (Ninth Series) and New Double Stars. By the Rev. T. E. Espin, M.A.

Name.	R.A. 1880	Decl.	P.	D.	Mags.		Nights.	Date. 1911
	h m	° ′	°	″				
η Cassiopeiæ	0 41·7	+57 11	242·3	6·04			3	·046
Hu 533	1 39·4	50 31	179·3	2·47	9·1	9·3	3	·076
A.G.	51·7	50 1	355·7	13·85	9·1	9·3	3	·911
Espin 454*	1 51·9	50 1	210·0	8·72	8·6	10·3	3	·911
Σ 673	5 11·5	50 29	266·8	1·77	8·8	10·0	4	·049 AB
			159·6	14·02	C=	13·2	3	·055 AC (New)
+50° 1152	11·9	50 16	116·7	9·45	8·8	9·5	2	·046
Σ 685	5 14·5	50 21	312·1	2·17	8·8	9·7	4	·518 AB
			185·2	20·52	C=	13·2	3	·688 AC (New)
+50° 1338	6 32·8	50 44	28·0	10·22	9·0	12·2	2	·097 AB
			65·8	32·50	C=	9·1	2	·097 AC
+50° 1348	6 37·0	50 3	292·1	10·00	9·3	9·4	3	·115
+49° 1615	7 10·4	49 52	317·9	10·33	8·5	11·1	3	·161
+47° 1919	11 53·4	47 16	51·5	9·27	8·9	12·0	2	·350
h 2602	12 6·3	46 58	224·1	24·55	9·7	10·7	1	·367
+47° 2068	13 28·5	47 25	343·5	9·40	8·6	12·2	3	·392
β 802	13 43·8	48 57	220·6	3·72	8·0	11·1	3	·429
β 1272	14 13·7	49 18	131·4	1·32	8·6	10·2	2	·448 AB
			321·0	23·60	C=	9·0	2	·448 AC
β 615	17·9	49 4	234·1	2·42	8·9	10·1	2	·448
Σ 1843	20·2	48 23	187·9	20·12	7·5	8·5	3	·418
OΣ 284	36·1	49 13	99·0	7·00	7·1	11·3	3	·425
Σ 1909	14 59·8	48 7	243·2	4·05	5·0	5·4	3	·413
Σ 1920	15 6·8	47 18	289·2	19·37	9·0	9·0	3	·424
Espin 624*	8·8	47 17	226·4	2·75	10·0	10·4	4	·424 AB
			133·9	18·54	C=	11·5	4	·424 AC (h 2770)
			128·0	18·80			3	·428 BC
Ku 50	22·9	46 35	333·8	3·24	9·1	9·4	3	·447
β 944	15 25·6	48 8	126·8	11·07	6·5	11·8	3	·447 AB
			67·7	56·02	C=	12·5	3	·447 AC
A.G. 201	16 0·2	49 17	252·5	7·56	9·0	9·1	4	·418
A.G. 202	2·0	47 59	283·8	22·37	9·1	9·2	3	·447
OΣ 307 rej.	7·0	48 7	201·4	17·73	7·0	10·5	3	·447
Σ 2047	16 19·7	47 54	327·3	2·20	8·0	8·2	4	·442
Σ 2133	17 5·6	49 55	197·0	3·20	8·9	9·4	3	·552
Σ 2142	8·6	49 53	114·6	5·19	5·7	9·3	3	·552
Σ 2153	17 14·8	+49 26	266·9	1·70	9·0	9·1	3	·552

Name.	R.A. 1880 h m	Decl. ° ′	P. °	D. ″	Mags.	Nights.	Date. 1911	
Σ 2210	17 39·1	+49 3	121·9	3·40	8·8 9·8	3	·552	
Σ 2251	17 51·5	49 39	26·7	14·98	8·6 10·1	3	·552	
Σ 2277	18 0·1	48 28	121·2	26·78	6·5 8·5	3	·606	AB
			286·2	88·98	C= 9·8	3	·606	AC
Σ 2293	6·8	48 22	82·4	12·77	8·1 9·3	2	·620	
+48° 2703	18 26·6	48 46	100·1	9·45	9·2 13·3	3	·631	
+49° 2929	19 2·1	49 44	129·7	12·31	6·7 12·0	3	·612	
h 1418*	30·0	49 46	10·9	29·20	9·1 9·6	3	·635	AB
			319·8	21·82	a= 13·0	3	·625	Aa
			312·8	18·30	b= 13·7	3	·635	Bb
+49° 3131	50·1	49 20	251·7	7·97	8·2 11·5	2	·639	
+48° 2979	19 50·6	48 56	305·0	16·87	7·5 10·5	2	·672	
Σ 2648	20 6·9	49 28	113·0	6·77	7·5 8·5	3	·634	
+49° 3216	7·6	49 33	312·3	9·20	8·2 12·1	3	·634	
+49° 3251	15·4	49 15	110·7	31·17	8·3 9·4	2	·672	AB
			186·4	17·40	C= 12·0	1	·676	BC
+48° 3117	20·9	48 18	321·3	11·39	7·7 10·2	3	·804	
+49° 3319	31·0	49 32	147·6	10·77	9·0 9·6	2	·672	
Espin 91	38·4	49 47	185·8	4·50	9·5 9·8	3	·641	AB
			233·8	18·08	C= 9·7	3	·641	AC
Espin 94	20 45·8	49 41	76·8	2·53	9·3 10·5	3	·767	BC
Espin 96	21 0·7	50 0	251·4	7·97	8·2 11·5	2	·636	
A.G.	22 27·6	52 36	32·3	9·12	8·5 8·8	3	·703	
+51° 3433	22 33·4	51 45	165·1	29·15	8·5 9·1	2	·766	AB
			245·1	13·95	C= 11·8	2	·766	BC
Storey	23 3·9	50 49	152·5	6·85	9·8 10·1	2	·683	
+52° 3487	32·3	52 9	142·8	19·50	7·7 9·0	2	·795	
+50° 4187	23 49·2	+51 3	97·6	12·20	8·5 9·1	2	·840	AB
			235·7	17·32	C= 12·0	2	·840	AC

Notes.

Espin 454. In the measures of 1907 the position angle should be 212°·2.

Espin 624. The star was measured in both 1908 and 1911 in twilight, and from its faintness was therefore a somewhat difficult object. The angle in 1911 is reversed—AC is *h* 2770, P 148°·4, D 14″±. It is possible that the change may be due to P.M.

h 1418. The minute *comites* escaped me when the star was measured in 1908.

New Double Stars.

No.	B.D.	R.A. 1900	Decl.	P.	D.	Mags.	Nights.	Date. 1911
		h m	° ′	°	″			
1052	+51° 74	0 21·1	+51 53	131·1	5·81	8·7 11·5	3	·868 (1910)
1053	+49° 252	0 52·3	49 22	80·5	5·45	9·1 9·8	2	·860
1054	+49° 338	1 10·8	50 11	62·3	5·13	9·4 11·7	3	·889
1055	+52° 419	37·2	53 4	261·4	7·35	9·2 12·0	3	·059
1056	+50° 347	39·5	50 40	20·3	4·80	9·5 9·7	2	·074
1057	+50° 345	39·3	50 51	5·3	7·88	9·5 13·9	3	·328
1058	+50° 405	50·9	50 26	50·8	5·22	9·0 12·0	2	·061
1059	+50° 417	53·2	50 48	219·8	8·45	8·7 11·0	2	·070 AB
				214·9	19·62	C=12·5	2	·070 AC
1060		1 57·5	50 22	252·5	2·90	9·6 11·5	2	·065
1061	+49° 589	2 7·1	49 57	113·0	3·25	9·5 9·6	2	·996
1062	+49° 590	7·5	50 9	28·3	7·03	9·5 9·7	3	·998
1063	+50° 568	2 24·3	50 37	25·1	2·05	9·5 11·0	2	·050
1064	+49° 1042	3 44·3	49 29	28·4	6·12	9·0 12·2	3	·366 BC
				287·8	59·35	A= 8·6	2	·131 AB
1065	+49° 1043	44·5	49 21	17·3	6·48	9·5 11·1	3	·003 (1912)
1066	+52° 743	3 54·9	52 47	112·3	5·70	9·3 9·5	2	·020
1067	+52° 773	4 1·8	52 41	41·8	4·72	9·4 12·0	2	·034
1068	+50° 962	9·2	51 3	72·4	3·60	13·0 13·5	2	·050 BC
				102·8	47·67	A= 8·7	2	·050 AB
1069	+49° 1171	13·7	49 26	109·1	4·13	9·2 13·2	3	·010 (1912)
1070	+49° 1263	4 43·9	49 57	112·2	2·23	9·4 10·6	3	·700
1071	+50° 1175	5 20·0	50 4	354·7	5·57	8·6 11·6	2	·016
1072	+49° 1366	5 25·2	49 8	359·5	4·05	9·3 10·4	2	·080
1073	+49° 1457	6 1·3	49 7	105·4	7·65	8·6 12·5	3	·703
1074	+49° 1476	9·8	49 42	179·6	6·27	9·5 11·3	2	·152 AB
				307·4	13·02	C= 10·5	2	·152 AC
1075	+52° 1104	30·1	52 32	3·6	6·52	9·4 12·0	2	·059
1076	+50° 1343	35·4	50 9	260·2	5·24	8·5 11·8	3	·145
1077		35·7	50 46	95·9	3·75	9·6 11·0	3	·124
1078	+52° 1154	53·4	52 12	285·5	4·51	9·3 11·7	4	·071
1079	+50° 1383	6 54·8	50 48	324·4	6·20	8·5 12·0	3	·115
1080	+49° 1599	7 4·5	49 40	29·5	4·62	9·5 10·6	2	·164
1081	+51° 1301	7·0	51 52	200·4	4·50	9·5 12·7	3	·128 AB
				99·6	37·45	C= 10·5	2	·115 AC
1082	+49° 1664	7 36·7	49 47	318·3	6·05	9·3 9·6	2	·176
1083	+49° 1795	8 54·4	49 6	355·5	5·85	9·3 9·4	2	·164
1084	+48° 1963	11 38·6	+47 50	257·7	4·17	9·5 10·7	2	·323

No.	B.D.	R.A. 1900 h m	Decl. ° ′	P. °	D. ″	Mags.		Nights.	Date. 1911	
1085	+47° 2129	14 12·7	+47 1	171·8	6·07	8·6	11·6	3	·369	
1086	+46° 2019	14 59·2	46 17	265·8	2·20	9·7	10·3	2	·428	
1087	+48° 2328	15 41·0	47 55	168·9	6·67	9·5	11·5	2	·414	
1088	+47° 2319	16 11·0	47 51	217·9	7·77	11·4	12·5	3	·492	BC
				316·4	33·37	A=	8·0	3	·492	AB
1089	+48° 2436	43·7	48 9	141·0	1·98	9·1	9·7	3	·525	AB
				30·1	32·57	C=	12·2	3	·525	AC
1090	+48° 2468	16 57·8	48 26	276·6	5·17	9·4	9·4	2	·523	
1091	+48° 2521	17 25·9	48 27	307·2	5·82	9·5	12·1	2	·527	
1092	+49° 2694	17 45·9	49 43	21·7	4·15	9·5	9·9	2	·513	
1093	+49° 2912	18 58·0	49 19	309·4	2·87	9·5	9·6	3	·652	
1094	+49° 2934	19 4·3	49 57	117·2	4·57	9·4	12·0	2	·658	
1095	+49° 2978	16·3	49 41	134·1	6·50	9·1	11·7	3	·612	
1096	+49° 3016	24·9	49 53	147·1	6·40	9·5	12·0	3	·621	
1097	+49° 3025	27·5	49 19	226·4	1·12	9·4	9·6	4	·664	
1098		19 42·2	48 34	165·9	3·82	9·3	9·8	3	·663	
1099	+48° 3053	20 7·6	49 5	179·6	4·62	9·0	10·0	2	·677	
1100	+49° 3299	27·4	49 54	281·6	2·30	9·5	9·6	2	·622	
1101		20 28·1	50 2	241·4	1·67	9·5	9·7	3	·635	
1102	+47° 3399	21 21·8	47 52	107·5	3·56	12·0	12·8	3	·864	CD
				49·6	5·02	C=	11·3	3	·864	BC
				194·9	46·71	A=	9·5	3	·864	AB
1103	+50° 3399	36·3	50 13	240·9	6·12	9·3	12·5	2	·703	
1104	+49° 3591	36·8	50 3	353·9	3·30	9·5	13·0	3	·932	AB
				225·3	13·90	C=	12·5	3	·932	AC
1105	+49° 3629	45·1	50 7	295·4	7·32	9·0	12·7	4	·917	AB
				107·7	11·00	C=	13·2	3	·907	AC
1106	+51° 3189	52·3	51 44	9·4	4·77	9·4	10·2	2	·746	AB
				158·2	10·77	C=	12·0	3	·771	AC
1107	+51° 3190	52·5	51 55	352·2	6·27	9·3	12·0	3	·772	
1108	+50° 3510	55·3	50 44	251·6	5·67	9·5	11·0	3	·643	
1109	+50° 3535	21 59·3	51 2	105·5	5·42	9·5	11·5	3	·661	
1110	+50° 3551	22 1·7	50 39	9·5	2·38	9·0	9·2	3	·906	
1111	+51° 3261	4·8	51 54	272·0	7·27	9·5	11·6	2	·828	
1112	+50° 3578	4·8	51 3	230·8	1·23	9·1	9·2	3	·672	
1113	+50° 3626	11·0	50 40	219·9	6·52	8·6	9·0	3	·976	
1114	+51° 3322	14·5	51 31	256·9	5·37	9·1	9·3	2	·971	
1115	+51° 3392	25·3	51 18	231·3	1·89	9·5	9·7	4	·819	
1116	+51° 3410	28·3	51 25	132·3	7·22	9·0	9·3	2	·803	
1117	+50° 3781	22 35·5	+50 42	268·6	5·67	9·5	12·5	2	·860	

No.	B.D.	R.A. 1900		Decl.		P.	D.	Mags.		Nights.	Date. 1911	
		h	m	°	′	°	″					
1118	+51° 3468	22	43·3	+51	39	131·6	6·67	9·5	11·3	2	·672	
1119	+51° 3509		52·7	51	36	115·8	4·51	9·4	9·8	2	·703	
1120	+50° 3893		53·8	51	1	23·2	7·97	9·3	9·4	3	·774	
1121	+51° 3524	22	56·9	52	7	331·9	1·18	9·3	9·5	3	·655	
1122	+50° 3940	23	1·7	50	38	83·5	2·50	9·2	9·3	2	·891	
1123	+51° 3565		10·4	52	2	309·6	2·53	9·4	10·0	3	·691	AB
						125·9	17·37	C=12·3		3	·691	AC
1124	+50° 4164		45·3	50	41	246·7	2·32	9·2	9·4	3	·931	
1125	+50° 4184	23	47·4	+51	6	333·8	5·05	9·5	9·6	3	·838	

Notes.

1077. The star B.D. + 50°1344 was found to be 13 mag. on Jan. 31, and is not identical with the pair here measured.

1121. Professor Fox kindly measured this object and obtained:—

1911·709 P 329°·4 D 1″·34 3 nts.

The Absorption of Light in Space. By Francis G. Brown.

(*Communicated by S. A. Saunder.*)

For some time past I have devoted some attention to the problem of the absorption of light in space. The researches of M. Tikhoff, Professor Turner, Professor Kapteyn, and others have indicated that some absorption exists, but the results have not been regarded as conclusive. It occurred to me that it might be possible to use the nebulæ in a determination of the extent of this absorption. Unlike the fixed stars, the nebulæ have a measurable diameter, and it is therefore possible to ascertain the relative distances of various groups of nebulæ, since it follows that those objects with a small apparent diameter must be, *on the average*, more distant than the larger objects, in spite of the fact that their real diameters vary greatly. If there were no absorption of light in its passage through space, the intrinsic, or surface, brightness of a nebula would be the same whatever its distance from the Earth; on the other hand, if the light were partly absorbed, the average brightness of the small nebulæ would be less than that of the larger ones.

The method of classification adopted in my investigations was as follows: using the descriptions in the New General Catalogue of Nebulæ, which are sufficiently exact when a large number of objects are dealt with, I separated the various classes marked

therein as "very large," "large," etc., and for the described brightness, substituted a number according to the following scale :—

eB = 9	pB = 5
vB = 8	pF = 4
B = 7	F = 3
cB = 6	vF = 2
eF = 1	

Classifying all the nebulæ down to those marked "very small" in the New General Catalogue, the following results are arrived at :—

TABLE I.

Average Brightness of the Nebulæ.

Class.	Approximate Diameter.	No. of Objects.	Total of the Numerical Substitutions for Brightness.	Average for Class.
vL	8′–10′	156	800	5·13
L	4′	325	1359	4·18
cL	3′	164	671	4·09
pL	1′–2′	832	2917	3·50
pS	30″–1′	998	2966	2·98
S	20″–30″	1975	5379	2·72
vS	10″–12″	1325	2609	1·97

One considerable disadvantage in the employment of this method arises from the fact that the existing catalogue brightness is not always reliable, owing to a tendency to over-estimate the brightness of a large diffused nebulæ, and to under-estimate that of a small condensed object. The differences between successive classes in the above table are, however, so striking, that it is evident that the greater part of the decrease is real. With more accurate data as to the brightness of the nebulæ, it should be possible to obtain a reliable determination of the amount of absorption for each class.

In the above classification I kept each hour of Right Ascension separate, and the results studied in this way are interesting. Table II. shows the average brightness of the various classes, together with certain combinations of classes over each of the twenty-four hours of right ascension. It will be seen that there is considerable variation in the brightness in successive hours, and this is noticeable not only in the case of the large nebulæ, which, being comparatively few in number, naturally show certain fluctuations, but also in the case of the smaller classes. If we calculate the harmonics for the groups vL – pL and pS – S, which are, of course, quite independent, we get the following, ϕ denoting the R.A. counted from the middle of group I. as zero : *i.e.* from R.A. $0^h\ 30^m$.

For vL − pL : $+0\cdot38 \sin\phi - 0\cdot14 \cos\phi + 0\cdot29 \sin 2\phi + 0\cdot04 \cos 2\phi + 0\cdot11 \sin 3\phi - 0\cdot36 \cos 3\phi$

For pS − S : $+0\cdot25 \sin\phi - 0\cdot26 \cos\phi - 0\cdot13 \sin 2\phi + 0\cdot01 \cos 2\phi - 0\cdot13 \sin 3\phi - 0\cdot30 \cos 3\phi$

TABLE II.

Mean Brightness of the various classes of Nebulæ in Right Ascension.

Hour.	vL.	L.	cL.	pL.	pS.	S.	vS.	vL − pL.	pS − S.	vL − S.	vL − vS.
I.	5·57	4·67	2·16	3·08	2·82	2·34	1·80	3·62	2·47	2·78	2·49
II.	5·20	4·08	4·14	3·65	1·92	2·58	2·02	3·87	2·41	2·89	2·63
III.	7·25	4·00	3·83	3·00	2·25	2·51	1·79	3·67	2·42	2·73	2·45
IV.	4·50	5·23	5·11	3·73	3·33	2·55	1·82	4·40	2·77	3·28	2·90
V.	6·57	5·64	5·00	3·44	3·31	2·86	1·73	4·56	3·03	3·52	3·07
VI.	6·33	4·91	3·75	3·53	2·88	3·50	3·30	5·19	3·30	3·69	3·66
VII.	—	4·00	3·25	3·59	3·36	2·81	1·85	3·57	2·98	3·23	2·91
VIII.	5·50	1·75	3·50	4·00	2·40	3·06	1·64	3·61	2·91	3·11	2·68
IX.	5·67	3·37	3·14	3·14	2·48	2·65	1·75	3·37	2·60	2·84	2·60
X.	5·33	5·00	4·20	2·90	3·07	2·62	2·04	3·74	2·78	3·08	2·88
XI.	3·85	4·64	5·00	3·24	2·83	2·79	2·07	3·94	2·80	3·22	3·05
XII.	5·86	4·33	5·22	4·22	3·60	3·00	1·96	4·57	3·22	3·70	3·27
XIII.	5·35	4·24	4·50	3·89	3·55	3·08	2·24	4·26	3·24	3·69	3·48
XIV.	4·90	3·92	4·22	3·83	3·25	2·94	2·14	3·97	3·05	3·39	3·14
XV.	3·25	3·94	3·60	3·80	3·03	2·50	1·96	3·79	2·70	2·96	2·79
XVI.	3·50	2·80	5·25	3·39	2·89	2·67	1·90	3·53	2·76	2·97	2·71
XVII.	2·50	5·50	3·75	3·15	1·92	2·24	1·84	3·35	2·13	2·48	2·23
XVIII.	5·00	3·28	2·00	3·09	2·37	2·35	1·86	3·38	2·35	2·65	2·41
XIX.	4·25	3·50	6·00	3·35	2·41	2·85	2·48	3·62	2·67	2·92	2·80
XX.	8·00	2·00	4·00	3·50	2·14	3·37	1·89	3·33	2·92	3·04	2·89
XXI.	2·33	4·00	4·25	3·24	2·59	2·94	1·85	3·41	2·81	3·04	2·79
XXII.	1·00	4·50	3·00	3·20	4·00	2·46	1·76	3·28	3·24	3·25	2·93
XXIII.	3·67	3·62	4·67	2·58	2·92	2·48	1·98	2·95	2·63	2·72	2·49
XXIV.	—	3·62	3·69	2·80	3·11	2·65	1·89	3·19	2·80	2·89	2·60

The second harmonics are discordant, but there is fair accord between the first and third.

As it appeared improbable that the variation depended directly on the right ascension, I then divided the sky into 88 areas, distributed as follows :—

8	in the zone between	N.P.D.	0°	and	30°	each	3 hrs.	of R.A.	in extent.
12	,,	,,	,,	30°	,,	55°	,,	2	,,	,,
24	,,	,,	,,	55°	,,	90°	,,	1 hr.	,,	,,
24	,,	,,	,,	90°	,,	125°	,,	1	,,	,,
12	,,	,,	,,	125°	,,	150°	,,	2 hrs.	,,	,,
8	,,	,,	,,	150°	,,	180°	,,	3	,,	,,
88										

Table III. shows the mean brightness of the nebulæ in each of the above areas. All the objects in the catalogue were made use of, with the exception of open and globular clusters, and nebulous stars. Where there are less than twenty objects in one area the results for two adjoining regions have been combined.

The distribution of the maximum and minimum regions may be seen in the diagram. There are a number of definite minima shown, and if we travel outward in any direction from these, the average brightness of the nebulæ increases until the zone affected by the next minimum is reached.

It is suggested by this diagram that the phenomenon is due to some physical cause, and not to accidental variation. It remains to be determined whether the variation represents a greater absorption of light in certain parts of the sky, or merely a difference in the actual brightness of the nebulæ in various regions. This may be tested by the star density. If a variable absorption exists, the stars in the regions where the average brightness of the nebulæ is at a minimum will appear more sparsely distributed than elsewhere. I have ascertained the star density near the points of minimum brightness of the nebulæ, and compared it with the density at points with the same galactic latitude where the brightness is approximately at a maximum. The counts made use of are those of Sir William Herschel * for the northern hemisphere, and those of Sir John Herschel † for the southern. The results are shown in Table IV., where the numbers inserted in the columns referring to star density represent the number of stars visible in a field of fifteen minutes diameter with Herschel's twenty-foot reflector.

The evidence of these star counts is in favour of the supposition that there is an increased absorption in the regions indicated on the diagram. Much useful information might be obtained from a comparison of the spectra of stars of similar type in those regions where a maximum or minimum degree of absorption is indicated. Meantime it may help further investigation to reduce Table III. to spherical harmonics, as in the next paper.

* Series I., Gauges 1-683, *Phil. Trans.*, 1785, p. 20. Reprinted by Holden, Publications of the Washburn Observatory, vol. ii., 1883.
Series II., Gauges 684–1088. Reduced by Holden as above.
† *Cape Results*, pp. 375–379.

TABLE III.—*Mean Brightness of the Nebulæ in 74 Areas.*

The upper figures refer to the total brightness, and the lower to the number of nebulæ in the area. Where the number of nebulæ is less than twenty in an area the results for two adjoining regions are combined.

Hour.	N.P.D. 0°–30°.		30°–55°.		55°–90°.		90°–125°.		125°–150°.		150°–180°.	
I.	5/2	Grouped with Hours 22–24.	55/14	2·96	334/147	2·27	335/141	2·37	70/32	2·58	73/23	3·07
II.	0		81/32		488/194	2·47	245/105	2·33	121/42		52/17	
III.	4/2		151/61	2·43	278/124	2·24	405/176	2·30	50/16	3·85	4/2	
IV.	10/3		70/30		42/19		406/164	2·47	196/48		29/8	
V.	16/5		3/1		51/21	2·87	315/122	2·58	154/38	3·68	176/48	3·66
VI.	6/2	3·13	2/1	2·57	38/10		86/35	2·45	41/15		700/192	
VII.	23/7		30/14		39/17	2·09	95/29	3·31	18/7	3·17	88/31	2·90
VIII.	29/9		96/35		126/62		44/13		4/1		25/8	
IX.	76/25		109/40	2·95	201/89	2·26	101/41	2·46	14/5		6/2	
X.	111/29		148/47		349/147	2·38	291/106	2·76	24/6		22/7	
XI.	105/34	3·51	192/61	3·59	309/156	2·50	254/104	2·44	86/23	3·71	13/3	3·55
XII.	135/37		397/103		856/290	2·95	267/105	2·54	44/12		22/7	
XIII.	142/38		341/96	3·49	1689/478	3·53	415/136	3·05	114/39	4·16	0	
XIV.	91/27	3·28	405/118		408/164	2·49	362/122	2·97	104/25		7/1	
XV.	33/16		322/107	2·98	426/163	2·62	155/56	2·77	27/9	3·04	11/4	2·50
XVI.	34/12		109/51		227/97	2·34	65/25	2·60	0		6/2	
XVII.	63/29	2·12	146/72	1·93	131/64	2·04	11/5	4·08	24/7		6/3	
XVIII.	109/56		99/55		118/50	2·36	95/21		22/8		27/11	
XIX.	64/27		54/25	2·37	126/48	2·62	44/13	3·17	27/8	3·23	45/16	3·00
XX.	14/9	2·18	36/13		42/12	2·98	29/10		70/22		23/9	
XXI.	14/6		17/6		104/37		79/37	2·13	88/28	3·61	34/9	
XXII.	9/4		14/4	2·58	99/46	2·15	110/47	2·34	179/46		26/7	
XXIII.	7/1	2·54	51/24		276/131	2·10	172/66	2·61	74/25	3·37	28/7	3·31
XXIV.	3/2		16/4		50/25	2·35	165/79	2·09	88/23		19/8	

AVERAGE BRIGHTNESS OF THE NEBULÆ IN 74 AREAS.

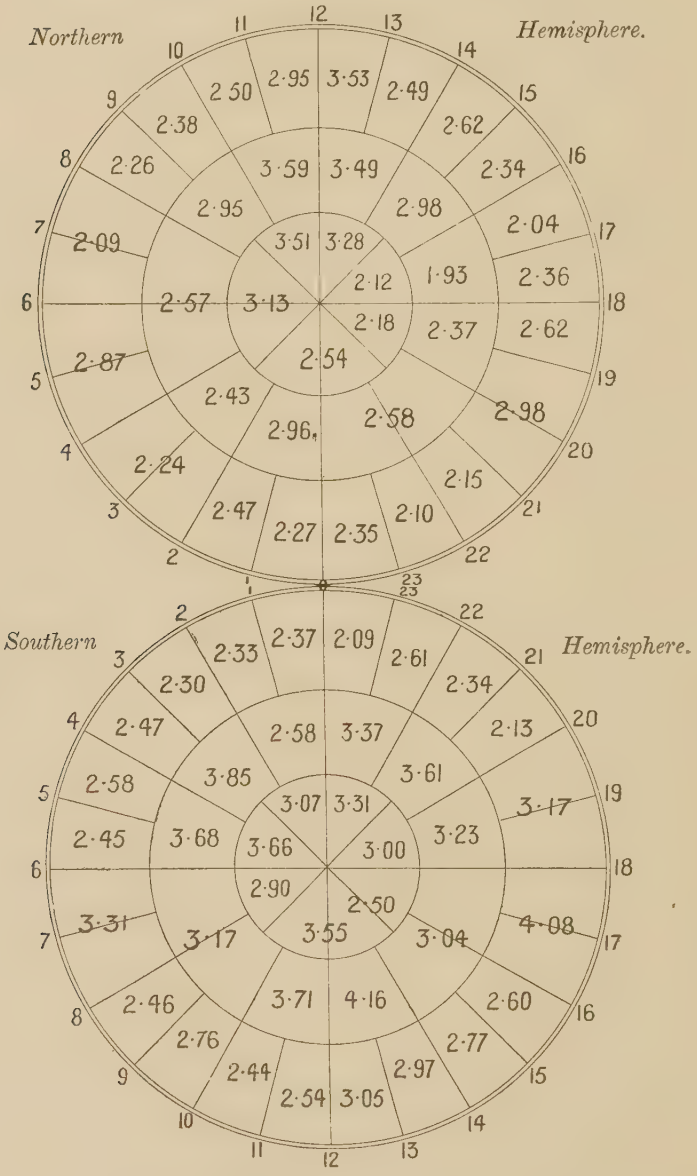

TABLE IV.

Comparisons of Star Density in Areas of Maximum and Minimum Brightness of the Nebulæ with the same Galactic Latitude.

N.B.—H refers to counts of Sir William Herschel ; *h* to those of Sir John Herschel.

Areas of Minimum Brightness.						Areas of Maximum Brightness.					
Position of Area.		Star Density.		Mean Star Density of Area.	Mean Galactic Latitude of Counts.	Position of Area.		Star Density.		Mean Star Density of Area.	Mean Galactic Latitude of Counts.
R.A.	Dec.	Gauge.	No. of Stars.			R.A.	Dec.	Gauge.	No. of Stars.		
h m	°					h m	°				
2·30	+0	H 53	4·9			12 0	+70	H 911	18·2		
		56	6·4					913	10·0		
		58	7·8					920	11·2		
		59	7·2					947	9·3		
		60	4·3								
		69	6·0								
		71	5·6								
		72	6·6								
		75	9·0								
		705	7·5			12·30	−10	H 924	9·6		
		709	7·4					926	10·3		
		77	6·8					931	7·3		
			——	6·6	50°				——	10·8	50°
7·30	+15	H 168	50			6·10	+0	H 137	44		
		179	60					138	60		
		197	48					140	57		
			——	52·0	10°			148	90		
								150	52		
								153	54		
								162	77		
								146	24		
								165	72		
									——	59·0	10°
10·0	+0	H 219	13·8			10−12	+80	H 890	60		
		223	11·5					902	25		
		226	7·9					904	11		
		230	7·2					911	18		
		231	6·5					913	10		
		232	4·9						——	25·0	42°
			——	8·6	37°						

TABLE IV.—*continued.*

Areas of Minimum Brightness.						Areas of Maximum Brightness.					
Position of Area.		Star Density.		Mean Star Density of Area.	Mean Galactic Latitude of Counts.	Position of Area.		Star Density.		Mean Star Density of Area.	Mean Galactic Latitude of Counts.
R.A.	Dec.	Gauge.	No. of Stars.			R.A.	Dec.	Gauge.	No. of Stars.		
h m	°					h m	°				
17·0	+35	H 316	14·2			10–12	+80	H 890	60		
		330	15·6					902	25		
			——	14·9	35°			904	11		
								911	18		
								913	10		
									——	25·0	42°
23·15	+ 0	H 669	7·5			11·0	+62	H 911	18·2		
		1071	6·3					932	8·0		
		1072	7·2					949	8·4		
		1081	7·1								
		1083	7·2			12–30	– 4	946	7·6		
			——	7·1	56°				——	10·5	60°
21·0	–15	*h*	14			21·0	–50	*h*	27		
		,,	11					,,	25		
		,,	12					,,	15		
			——	12·3	30°			,,	17		
									——	21·0	30°
5·30	–10	*h*	9			5·0	–50	*h*	11		
		,,	18					,,	9		
		,,	12					,,	17		
		,,	13					,,	17		
			——	13·0	38°			,,	20		
								,,	21		
								,,	9		
								,,	12		
									——	14·5	38°
1·0	–48	*h*	7			3·0	– 44		2		
		,,	4						10		
		,,	6						10		
		,,	10						6		
		,,	1						12		
		,,	11						13		
			——	6·5	67°				6		
									——	8·4	67°
		MEANS :—		10·4	40°					21·8	42°

An example of the use of Spherical Harmonic Analysis.

By H. H. Turner, D.Sc., F.R.S., and F. G. Brown.

1. The use of simple harmonic analysis is only slowly making its way into astronomical computations, although there are many departments in which it is specially suitable, *e.g.* in the observations of variable stars. Even in gravitational astronomy, where the theory and tables have long been expressed in terms of sines and cosines, the observations have seldom been analysed to correspond. Dr. Cowell's analysis of the Greenwich lunar observations is a recent example, but was almost the first.

2. Hence it is scarcely matter for surprise that the rather more elaborate spherical harmonic analysis has made so little headway up to the present. And yet there is special need of it in astronomy for expressing compendiously various distributions over the sphere. Thus we might express the number of stars to the 5th magnitude in this way; and then to the 6th; and to the 7th; and so on: and the run of the coefficients would give us at a glance important information. Presently we may hope to express distributions of proper motions by this method. Again there is the distribution of nebulæ, which can doubtless be shown on a diagram in various ways, but could also be exhibited in a numerical series. Both methods have advantages, and the use of either need not imply disparagement of the other.

3. It seems possible that if the method of spherical harmonic analysis were put into simple and accessible shape, it might be used by some who have hitherto been repelled by the complex appearance of some of the formulæ. The following example is given at some length with this object in view, while at the same time it serves the special purpose of bringing out the main features of the distribution of brightness of nebulæ in different parts of the sphere, given in Table III. of the preceding paper by one of us.

4. On reference to this table it will be seen that the results are collected in zones of N.P.D. (or declination), as will usually be the case in astronomical work. But to avoid seeming to make this necessary, we will use θ for N.P.D., and ϕ for R.A. The first step is to express the results for *each zone separately* in ordinary harmonics; that is, to express each zone in the form

$$a_0 + a_1 \cos\phi + b_1 \sin\phi + a_2 \cos 2\phi + b_2 \sin 2\phi + a_3 \cos 3\phi + a_3 \sin 3\phi + \text{etc.}$$

We must at this point determine how many orders of harmonics we are going to use, and how many figures; for this will characterise all the subsequent work. Experience suggests that at any rate in the first instance we should use *three orders only* (so far as $\cos 3\phi$ and $\sin 3\phi$), and should use *two decimal places* only. In some cases the accordance of the residuals will suggest that we might go further, but little will have been lost by the preliminary work, which will serve as a useful check on the more elaborate investi-

gation; and in many cases the residuals will show that we have got all we can with *two decimal places*.

5. It is unnecessary to explain at any length the method for obtaining the coefficients of $\cos\phi$, $\sin\phi$, etc., for the different zones, but it may be remarked that the observations should, if possible, be grouped so as to have centres at 0^h, 2^h, 4^h . . . 22^h, since this simplifies the computations. The values of $\cos\phi$ and $\sin\phi$ are then

$\cos\phi$	$+1$	$+f$	$+\frac{1}{2}$	0	$-\frac{1}{2}$	$-f$	-1	$-f$	$-\frac{1}{2}$	0	$+\frac{1}{2}$	$+f$
$\sin\phi$	0	$+\frac{1}{2}$	$+f$	$+1$	$+f$	$+\frac{1}{2}$	0	$-\frac{1}{2}$	$-f$	-1	$-f$	$-\frac{1}{2}$

where f is written for sin 60° or ·866 or $1 - 1/10 - 1/30$. The use of tables, either of functions or for multiplying, is then unnecessary, since we can write down the operations at sight. The values of $\cos 2\phi$ and $\sin 2\phi$, $\cos 3\phi$ and $\sin 3\phi$, etc., are all selections from the above, and the operations tend to become simpler and simpler.

These twelve groups will usually suffice, but if it is desired to use twenty-four groups, the intermediate factors for 1^h, 3^h, 5^h, etc., may be written $1/4$, $1 - 3/10$, $1 - 1/30$, without sensible loss of accuracy.

6. We will assume, then, that the results of Table III. in the preceding paper have been analysed in this way, and that the coefficients found for each zone are as shown in Table I.

TABLE I.

Coefficients of the Harmonics in Mr. Brown's Zones.

Zone of N.P.D.	Mean Value.	Coefficients of $\cos\phi$.	$\sin\phi$.	$\cos 2\phi$.	$\sin 2\phi$.	$\cos 3\phi$.	$\sin 3\phi$.	A.
0°– 30°	2·81	−·38	+·45	+·19	−·05	−·11	−·09	·14
30– 55	2·75	−·34	+·14	+·46	−·03	−·12	−·06	·29
55– 90	2·49	−·13	−·07	+·05	+·04	−·28	−·05	·57
90–125	2·76	−·28	−·24	−·42	+·09	+·19	+·12	·57
125–150	3·38	−·13	+·08	+·10	+·02	−·46	−·17	·29
150–180	3·15	+·03	+·27	+·21	−·08	−·33	−·04	·14

The second column is simply the mean value for the zone, which varies only with θ, and must be expressed by terms in θ only. The next column must be expressed by spherical harmonics which all contain $\cos\phi$ as a factor and no other term in ϕ; the next by $\sin\phi$ terms, etc. In the last column of Table I. is given twice the area of the zone as a fraction of that of the whole sphere. By the well-known properties of the sphere these numbers are simply $(\cos\theta_2 - \cos\theta_1)$ where θ_1 and θ_2 denote the N.P.D.'s of the limits of the zone. We shall want this column in the work.

7. We have now accordingly to express each of the above columns (except the first and last) as a series of terms in θ. They will no longer be simple sines and cosines of θ and its multiples,

but different functions, the analytical form of which is given at the end of the paper. But let us not concern ourselves with these theoretical forms at present; all we want is their numerical expression in the form of tables, and this is sufficiently provided for our present purpose in Table II. We will denote the expressions by the Greek letters α, β, γ, etc., and the different orders by 1, 2, 3, 4 . . ., so that we are to find for the columns of Table I. the equivalent expressions

for	"mean value"	$a_0 + a_1\alpha_1 + a_2\alpha_2 + a_3\alpha_3$
for	$\cos\phi$	$b_1\beta_1 + b_2\beta_2 + b_3\beta_3$
for	$\sin\phi$	$c_1\beta_1 + c_2\beta_2 + c_3\beta_3$
for	$\cos 2\phi$	$d_2\gamma_2 + d_3\gamma_3$
for	$\sin 2\phi$	$e_2\gamma_2 + e_3\gamma_3$
for	$\cos 3\phi$	$f_3\delta_3$
for	$\sin 3\phi$	$g_3\delta_3$

that is to say, we are to find the numerical coefficients a_0, a_1, a_2 . . . b_1, b_2 . . ., by a process very similar to (but not quite the same as) that used for simple harmonics, when the functions α, β, γ, . . . have the values shown in Table II. It will be seen that the work for $\cos\phi$ and $\sin\phi$ is much the same, and for $\cos 2\phi$ and $\sin 2\phi$, and so on. Also that we begin with higher orders each time, so that the work continually lessens.

TABLE II.

Values of the Functions of θ, *multiplied* 100 *times.*

θ.	α_1 or $\cos\theta$.	α_2 or $\frac{3}{4}\cos 2\theta + \frac{1}{4}$.	α_3 or $\frac{5}{8}\cos 3\theta + \frac{3}{8}\cos\theta$.	β_1 or $\sin\theta$.	β_2 or $\sin 2\theta$.	β_3 or $\frac{1}{6}\sin\theta + \frac{5}{6}\sin 3\theta$.	$\frac{1}{2} - \frac{1}{2}\gamma_2$ or $\cos 2\theta$.	γ_3 or $\frac{3}{4}\cos\theta - \frac{1}{4}\cos 3\theta$.	δ_4 or $\frac{3}{4}\sin\theta - \frac{1}{4}\sin 3\theta$.
°									
0	+100	+100	+100	0	0	0	0	0	0
10	+ 98	+ 96	+ 91	+ 17	+34	+45	+ 3	+ 8	0
20	+ 94	+ 83	+ 66	+ 34	+64	+78	+ 12	+ 29	+ 4
30	+ 87	+ 62	+ 33	+ 50	+87	+92	+ 25	+ 59	+ 12
40	+ 77	+ 38	− 2	+ 64	+98	+83	+ 41	+ 85	+ 26
50	+ 64	+ 12	− 30	+ 77	+98	+54	+ 59	+101	+ 45
60	+ 50	− 12	− 44	+ 87	+87	+14	+ 76	+100	+ 65
70	+ 34	− 33	− 41	+ 94	+64	− 26	+ 88	+ 80	+ 83
80	+ 17	− 45	− 25	+ 98	+34	− 56	+ 96	+ 45	+ 95
90	+ 0	− 50	0	+100	0	− 67	+100	0	+100
100	− 17	− 45	+ 25	+ 98	−34	− 56	+ 96	− 45	+ 95
110	− 34	− 33	+ 41	+ 94	−64	− 26	+ 88	− 80	+ 83
120	− 50	− 12	+ 44	+ 87	−87	+14	+ 76	−100	+ 65
130	− 64	+ 12	+ 30	+ 77	−98	+54	+ 59	−101	+ 45
140	− 77	+ 38	+ 2	+ 64	−98	+83	+ 41	− 85	+ 26
150	− 87	+ 62	− 33	+ 50	−87	+92	+ 25	− 59	+ 12
160	− 94	+ 83	− 66	+ 34	−64	+78	+ 12	− 29	+ 4
170	− 98	+ 96	− 91	+ 17	−34	+45	+ 3	− 8	0
180	−100	+100	−100	0	0	0	0	0	0

8. Now the property of spherical harmonics which we are to use is

$$\int h_1 h_2 d\mathrm{S} = 0$$

where h_1 and h_2 are any different harmonics, and $d\mathrm{S}$ is an element of area.

When the zones are symmetrical in the N. and S. hemispheres the products $h_1 h_2$ in many cases cancel out independently of the factor $d\mathrm{S}$. This is always the case when we multiply one spherical harmonic containing $\cos m\phi$ by another containing $\cos n\phi$ or $\sin n\phi$, by the properties of ordinary harmonics. But it is also sometimes true when we multiply together two harmonics containing the same function of ϕ, say $\cos m\phi$. The product thus contains the factor $\cos^2 m\phi$, which is always positive. But the θ factors may nevertheless make it cancel out independently of $d\mathrm{S}$. Thus, if we multiply together the second and third columns of Table II., we see that the first product $+ 100 \times 100$ is exactly balanced by the last product $- 100 \times 100$, and so on for every pair from the two ends. Similarly, the products of the third and fourth columns cancel out automatically, apart from reference to $d\mathrm{S}$. But if we multiply the second into the fourth as they stand we see that the upper half is repeated in the lower half, *not* with opposite sign as hitherto, but with the *same* sign; and each half has products as in the first two columns below:—

	As they stand.		Multiplied by $\sin\theta$.	
	+	−	+	−
	100	2	0	1
	89	19	15	15
	62	22	21	19
	29	14	14	13
		4		4
Sums	280	61	50	51

The positives are much in excess of the negatives because the zones near the poles have been given undue weight. An element of zone area is $2\pi \sin\theta d\theta$, and when we multiply each product by $\sin\theta$, as in the last two columns, we get a sensible balance between + and −. So also when we multiply the fifth column by the seventh. In other cases the omission of the factor $\sin\theta$ would merely give undue weight to the polar regions. It may, again, be remarked that many cases are cancelled out by the ϕ factors: we need not consider any intermultiplication of columns with different Greek letters at their head, for they have different ϕ factors and vanish accordingly.

9. Let us now begin with the first column of Table I., the mean values m, say, which are to be expressed in the form

$$m = a_0 + a_1\alpha_1 + a_2\alpha_2 + a_3\alpha_3.$$

We are to find a_1 by—

(1) multiplying all the numbers m by the proper value of $a_1 A$ and adding up. By what has been said the coefficients of a_2 and a_3 will cancel out and we shall get simply—

$$\Sigma m a_1 A = a_0 \Sigma a_1 A + a_1 \Sigma a_1^2 A.$$

And, although it has not yet been mentioned, the sum $\Sigma a_1 A$ will always be zero also. We must therefore—

(2) find the sum $\Sigma a_1^2 A$, and use it as a divisor of the previous sum to find a_1.

10. The other coefficients are all found in precisely the same way, including a_0; but we must remember that the appropriate factor for a_0 is not unity but A. We must not take the *simple* mean of the second column, but the weighted mean according to area of zone. We can, however, subtract any convenient constant before doing this—say the smallest member 2·49, and call the remainder m'. The whole process of finding the numerators is then as shown in Table III.

Table III.

Calculation of Numerators.

Mean θ.	m'.	A.	a_1.	a_2.	a_3.	Am'.	$Am'a_1$.	$Am'a_2$.	$Am'a_3$.
15°	+·32	·14	+·96	+·90	+·78	+·045	+·043	+·041	+·035
42½	+·26	·29	+·74	+·31	−·09	+·075	+·056	+·023	−·007
72½	·00	·57	+·30	−·36	−·38	·000	·000	·000	·000
107½	+·27	·57	−·30	−·36	+·38	+·154	−·046	−·055	+·059
137½	+·89	·29	−·74	+·31	+·09	+·258	−·191	+·080	+·023
165	+·66	·14	−·96	+·90	−·78	+·092	−·088	+·083	−·072
	Sums	2·00				+·624	−·226	+·172	+·038

The mean value of m over the surface of the sphere is thus $2{\cdot}49 + {\cdot}624/2{\cdot}00 = 2{\cdot}80$.

11. For the denominators we multiply the squares of $a_1\, a_2\, a_3$ by A and add them up; but since they repeat in the lower half, we need not perform the second series of multiplications if we use 2A instead of A.

Table IV.

Calculation of Denominators.

2A.	a_1^2.	a_2^2.	a_3^2.	Products.		
·28	·92	·81	·61	·258	·227	·171
·58	·55	·07	·01	·319	·041	·006
1·14	·09	·13	·14	·103	·148	·160
			Sums	·680	·416	·337

12. Hence the coefficients are :—

$$-\cdot226/\cdot680 = -0\cdot33 \text{ for } \cos\theta \qquad \text{or } a_1$$
$$+\cdot172/\cdot416 = +0\cdot40 \text{ for } (3\cos 2\theta + 1)/4 \qquad \text{or } a_2$$
$$+\cdot038/\cdot337 = +0\cdot11 \text{ for } (5\cos 3\theta + 3\cos\theta)/8 \text{ or } a_3$$

13. We can now see how far these fit the observations (indicated by the first column) by multiplying these coefficients into the numbers in Table III.

TABLE V.

Comparison with Observation for Terms in θ only.

θ.	$-\cdot33a_1$.	$+\cdot40a_2$.	$+\cdot11a_3$.	Sum.	$m-2\cdot80$.	$O-C_3$.	C_4.	$O-C_3-C_4$.
15°	−·32	+·36	+·09	+·13	+·01	−·12	−·23	+·11
42½	−·24	+·12	−·01	−·13	−·05	+·08	+·13	−·05
72½	−·10	−·14	−·04	−·28	−·31	−·03	−·02	−·01
107½	+·10	−·14	+·04	·00	−·04	−·04	−·02	−·02
137½	+·24	+·12	+·01	+·37	+·58	+·21	+·13	+·08
165	+·32	+·36	−·09	+·59	+·35	−·24	−·23	−·01

14. The residuals $O-C_3$ suggest the existence of a small harmonic of the fourth order, which has the value $(9+20\cos 2\theta + 35\cos 4\theta)/64$, and has the following values for our particular cases.

TABLE VI.

θ.	a_4.	$A(O-C_3)$.	$Aa_4(O-C_3)$.	a_4^2.	$2Aa_4^2$.
15°	+·69	−·017	−·012	·48	·134
42½	−·37	+·023	−·008	·14	·081
72½	+·07	−·017	−·002	·00	·000
107½	+·07	−·023	−·002		
137½	−·37	+·061	−·023		
165	+·69	−·034	−·023		
		Sums	−·070		·215

15. Hence coefficient of $(9+20\cos 2\theta + 35\cos 4\theta)/64$ is

$$-\cdot070/\cdot215 = -0\cdot33.$$

The values of this quantity are given under the heading C_4 in Table V., and the residuals are now smaller. But we must be careful in ascribing any physical significance to this fact; for we have already used four arbitrary constants to represent five quantities (the six observed zones, less one mean of all). We can, however, regard the smallness of the residuals and the character of their sequence as evidence of the correctness of the arithmetic; and it is noticeable that the fourth harmonic is greater than the third.

16. The values of other coefficients are all found in the same way, and without tedious repetition we may perhaps now give the results of calculation of all the harmonics to the third order.

TABLE VII.

Coeff.	Expression in (θ, ϕ).	Expression in (x, y, z).
−0·33	$\cos\theta$	$-\cdot33z$
−0·25	$\cos\phi \sin\theta$	$-\cdot25x$
−0·08	$\sin\phi \sin\theta$	$-\cdot08y$
+0·40	$(3\cos 2\theta + 1)/4$	$+\cdot40z^2 - \cdot20x^2 - \cdot20y^2$
−0·04	$\cos\phi \sin 2\theta$	$-\cdot08xz$
+0·08	$\sin\phi \sin 2\theta$	$+\cdot16yz$
−0·11	$\cos 2\phi \sin^2\theta$	$-\cdot11x^2 + \cdot11y^2$
+0·06	$\sin 2\phi \sin^2\theta$	$+\cdot12xy$
+0·11	$(5\cos 3\theta + 3\cos\theta)/8$	$+\cdot28z^3 - \cdot41x^2z - \cdot41y^2z$
−0·09	$\cos\phi(\sin\theta + 5\sin 3\theta)/6$	$-\cdot24xz^2 + \cdot06xy^2 + \cdot06x^3$
+0·29	$\sin\phi(\sin\theta + 5\sin 3\theta)/6$	$+\cdot80yz^2 - \cdot20xy^2 - \cdot20y^3$
+0·27	$\cos 2\phi(2\cos\theta - 2\cos 3\theta)/3$	$+\cdot72x^2z - \cdot72y^2z$
−0·03	$\sin 2\phi(2\cos\theta - 2\cos 3\theta)/3$	$-\cdot16xyz$
−0·11	$\cos 3\phi(3\sin\theta - \sin 3\theta)/4$	$-\cdot11x^3 + \cdot33xy^2$
+0·02	$\sin 3\phi(3\sin\theta - \sin 3\theta)/4$	$+\cdot06x^2y - \cdot02y^3$

Significance of these Expressions.

17. Most of the information likely to be collected at present in Astronomy will be collected in terms of R.A. and Decl., and probably arranged in zones of declination. The calculations of harmonics will thus be most conveniently made in terms of (α, δ) or (ϕ, θ). But once these are obtained, the discussion is most conveniently made in terms of x, y, z, since these coordinates can more readily be transformed to another system of axes.

What these axes shall be must be determined from the harmonics themselves.

Terms of the First Order.

18. The terms of the first order give us a diametral plane

$$33z + 25x + 8y = 0$$

or if we prefer it, an axis at right angles to this, whose direction cosines are given by

$$\frac{x}{25} = \frac{y}{8} = \frac{z}{33} = \frac{1}{\sqrt{25^2 + 8^2 + 33^2}} = \frac{1}{42}.$$

Thus $$\cos\theta = \frac{33}{42} = \cdot 83, \quad \theta = +36^\circ$$

$$\tan\phi = \frac{8}{25} = \cdot 32 \quad \phi = +18^\circ.$$

If we take the Pole of the Milky Way at

$$\theta = 117^\circ, \ \phi = 0^h\ 48^m = +12^\circ,$$

so that its direction cosines are

$$x' = +\cdot 87 \quad y' = +\cdot 19 \quad z' = -\cdot 45,$$

the angle between the axis and the polar axis of the Milky Way is

$$\cos^{-1}(25 \times \cdot 87 + 8 \times \cdot 19 - 33 \times \cdot 45)/42 = \cos^{-1} \cdot 20 = 78^\circ.$$

Considering the nature of the observations, we may take it that a point in the Milky Way is indicated for one end of the axis. The brightness accordingly increases systematically as we proceed from one end of this axis to the other; one possible interpretation of which is that our view is most obstructed in some way in this particular direction (roughly R.A. 1^h, Dec. $+60^\circ$).

Terms of the Second Order.

19. On collecting the various terms of the second order and multiplying by 100 (to avoid decimals) we get

$$-31x^2 - 9y^2 + 40z^2 + 16yz - 8zx + 12xy,$$

and we find the axes of this ellipsoid by solving the equation

$$\begin{vmatrix} \lambda - 31, & +6, & -4 \\ +6, & \lambda - 9, & +8 \\ -4, & +8, & \lambda + 40 \end{vmatrix} = 0$$

or $$\lambda^3 - 1440\lambda + 12000 = 0$$

where needless figures in the last places are dropped. It will be noticed by reference to the formulæ that the term in λ^2 vanishes automatically. Solving this equation in any convenient manner (trial is perhaps quickest) we find

$$\lambda = +33, \quad +8\cdot 5, \quad -41.$$

The root $\lambda = +33$ gives, on substituting in the separate equations,

$$\begin{aligned} 2x + 6y - 4z &= 0 \\ 6x + 24y + 8z &= 0 \\ -4x + 8y + 73z &= 0 \end{aligned}$$

or $$\frac{x}{36} = \frac{y}{-10} = \frac{z}{3} = \frac{1}{37}$$

or $$\phi = -16^\circ \quad \theta = 85^\circ.$$

The angle between this direction and the pole of the Milky Way is

$$\cos^{-1}(36 \times \cdot 87 - 10 \times \cdot 19 - 3 \times \cdot 45)/37 = \cos^{-1} \cdot 76 = 40^\circ.$$

This is not small. All that can be said is that this axis is nearer the Pole than the Galaxy itself.

20. The second root ($\lambda = +8 \cdot 5$) gives

$$\frac{x}{10} = \frac{y}{34} = \frac{z}{-7} = \frac{1}{36}$$

or $\quad \phi = 74^\circ \quad \theta = 101^\circ$

and the third root ($\lambda = -41$) gives

$$\frac{x}{-10} = \frac{y}{36} = \frac{z}{240} = \frac{1}{243}$$

or $\quad \phi = 106^\circ \quad \theta = 9^\circ.$

This last is the nearest to the axis indicated by the terms of the first order. The angle between them is

$$\cos^{-1}(-250 + 288 + 7920)/243 \times 42 = \cos^{-1} \cdot 78 = 39^\circ.$$

21. Is this angle small enough to be worth attention?

Let us suppose that we are given a set of axes: what is the chance that a point will fall within distance θ of any one of them? Describe little circles, radius θ, round each of the six ends: they include altogether an area $3(1 - \cos\theta)$ of the sphere, which for different values of θ is as follows

$\theta =$	5°	10°	15°	20°	25°	30°	35°	40°	45°
chance =	·02	·05	·10	·18	·28	·40	·54	·70	·88

After 45° the separate circles overlap.

Hence the connection of the second order terms with the Milky Way, or with the first harmonic, is not very clear.

22. Beyond the indications of the terms of the first order, which seem to have a connection with the Milky Way, nothing very definite comes out of these numbers at present. But their real use is for comparison with others of a cognate character; and since these are not yet in existence, the discussion must be incomplete.

23. In conclusion, we will tabulate the analytical expressions for the first few orders of spherical harmonics. The simplest form for *numerical calculation* is in terms of θ and ϕ; but the analytical expressions are simpler in terms of x, y, z, where

$$x = \sin\theta\cos\phi, \quad y = \sin\theta\sin\phi, \quad z = \cos\theta,$$

and we may give these first.

TABLE VIII.

Tabulation of the First Four Orders of Harmonics in Terms of (x, y, z).

Characteristic Factors.		Additional Factors.			
		1st Order.	2nd Order.	3rd Order.	4th Order.
I	I	z	$z^2-\frac{1}{2}r^2$	$z^3-\frac{3}{2}zr^2$	$z^4-3z^2r^2+\frac{3}{8}r^4$
$\cos\phi$	x	1	$2z$	$\frac{8}{3}z^2-\frac{2}{3}r^2$	$4z^3-3zr^2$
$\sin\phi$	y				
$\cos 2\phi$	x^2-y^2	...	1	$\frac{8}{3}z$	$8z^2-\frac{4}{3}r^2$
$\sin 2\phi$	$2xy$				
$\cos 3\phi$	x^3-3xy^2	...	...	1	$3z$
$\sin 3\phi$	$3x^2y-y^3$				
$\cos 4\phi$	$x^4-6x^2y^2+y^4$	...	...	...	1
$\sin 4\phi$	$4x^3y-4xy^3$				

In the first column of Table VIII. is given the function of ϕ which enters into the harmonic, the rest of it being made up entirely of functions of θ, as we shall see in Table IX. In the second column this factor is converted into x and y; thus

$$x^2-y^2=\sin^2\theta(\cos^2\phi-\sin^2\phi)=\sin^2\theta\cos 2\phi$$
$$2xy=\sin^2\theta(2\sin\phi\cos\phi)=\sin^2\theta\sin 2\phi.$$

The harmonics below the top line all occur in pairs, one member containing $\cos n\phi$ and the other $\sin n\phi$, and the factor in θ being the same for both. It will be seen that the expressions in the second column may be obtained by expanding the binomial

$$(x+y\sqrt{-1})^n=x^n \quad -\frac{n(n-1)}{1\cdot 2}x^{n-2}y^2 \quad +$$
$$+\sqrt{-1}\left\{nx^{n-1}y \quad -\frac{n(n-1)(n-2)}{1\cdot 2\cdot 3}x^{n-3}y^3+\ \ldots\ \right\}$$

the real part being the upper factor in Table II., and the imaginary part the lower factor.

The rest of the table consists of powers of z and r^2 only, where r^2 is written for x^2+y^2 to save space. Rules can be given for the formation of the coefficients, but if any are required beyond the fourth it is almost simpler to calculate them from the equation

$$\frac{d^2V}{dx^2}+\frac{d^2V}{dy^2}+\frac{d^2V}{dz^2}=0.$$

For example, if we want the harmonic of the fifth order multiplied by x^2-y^2, it must be of the form

$$C(x^2-y^2)\{z^3+az(x^2+y^2)\}$$

which on expansion is

$$V=C(z^3x^2-z^3y^2+azx^4-azy^4).$$

Now

$$\frac{d^2V}{dx^2} = C(2z^3 + 12azx^2)$$

$$\frac{d^2V}{dy^2} = C(-2z^3 - 12azy^2)$$

$$\frac{d^2V}{dz^2} = C(6x^2 - 6y^2)z$$

from which we see that $12a = -6$ or $a = -\frac{1}{2}$, and the harmonic required is

$$C(x^2 - y^2)\{z^3 - \tfrac{1}{2}z(x^2 + y^2)\}.$$

We have still to choose the constant C, and it really does not much matter what we write for it. We could put $C = 2$, so as to make all the coefficients integral, or $C = 20$. In the latter case we should find (in the course of the work on the observations) a coefficient just one-tenth of the former. But there is a convenience in having the harmonics all of *about* the same size; and we may, for instance, choose C so that the maximum value is unity, as has been done in the tables given (in which we accordingly depart from the values used by Professor Love). A theoretically preferable procedure would be to choose C so that the integrated value of the square of the harmonic over the sphere is unity; but this involves some little work, and the simpler method is all we require.

TABLE IX.

Expression of the same Harmonics in Terms of ϕ and θ.

Characteristic Factor.	Additional Factors. 1st Order.	2nd Order.	3rd Order.	4th Order.
1	$\cos\theta$	$(1+3\cos 2\theta)/4$	$(3\cos\theta+5\cos 3\theta)/8$	$(9+20\cos 2\theta+35\cos 4\theta)/64$
$\cos\phi$ / $\sin\phi$	$\sin\theta$	$\sin 2\theta$	$(\sin\theta+5\sin 3\theta)/6$	$(2\sin 2\theta+7\sin 4\theta)/8$
$\cos 2\phi$ / $\sin 2\phi$	...	$(1-\cos 2\theta)/2$	$(2\cos\theta-2\cos 3\theta)/3$	$(3+4\cos 2\theta-7\cos 4\theta)/8$
$\cos 3\phi$ / $\sin 3\phi$	...	...	$(3\sin\theta-\sin 3\theta)/4$	$(6\sin 2\theta-3\sin 4\theta)/8$
$\cos 4\phi$ / $\sin 4\phi$	...	...	...	$(3-4\cos 2\theta+\cos 4\theta)/8$

The rules for forming the additional factors are now a little more complicated, though they have of course been worked out. If the harmonic of order i which contains the factor $\cos n\phi$ or $\sin n\phi$ be denoted by $S\cos n\phi$ or $S\sin n\phi$, then S must satisfy Laplace's equation, which may be written now in the form

$$2(1-\cos 2\theta)\frac{d^2S}{d\theta^2} + 2\sin 2\theta \cdot \frac{dS}{d\theta} + 2\{i(i+1) - 2n^2 - i(i+1)\cos 2\theta\}S = 0.$$

We see by inspection of the above table that S will consist of a series of cosines of multiples of θ, or else of a series of sines. Let us take the cosines first, and suppose $p_m \cos m\theta$ is one of them.

As regards this term alone,

$$\frac{d^2S}{d\theta^2} = -p_m m^2 \cos m\theta \quad \text{and} \quad \frac{dS}{d\theta} = -p_m m \sin m\theta.$$

Since now the products

$$2 \cos m\theta \cos 2\theta = \cos \overline{m-2}\theta + \cos \overline{m+2}\theta$$

and

$$2 \sin m\theta \sin 2\theta = \cos \overline{m-2}\theta - \cos \overline{m+2}\theta$$

the term $p_m \cos m\theta$, when substituted in Laplace's equation, will be of the form

$$p_m(P_m \cos \overline{m+2}\theta + Q_m \cos m\theta + R_m \cos \overline{m-2}\theta)$$

where

$$P_m = m(m+1) - i(i+1)$$
$$Q_m = -2\{m^2 + 2n^2 - i(i+1)\}$$
$$R_m = m(m-1) - i(i+1).$$

If we had started with $\sin m\theta$, we should have got the same values for P, Q, R. From these the actual values of p_m can be deduced; but instead of using the actual formulæ, it is perhaps simpler to form the coefficients directly by equating to zero the coefficients of $\cos \overline{m+2}\theta$, $\cos m\theta$, and so on. An example will make this clear; take the same example as before: fifth order (so that $i=5$), and containing $\cos 2\phi$ (so that $n=2$). The term in $\cos 5\theta$ (or $m=5$) gives rise to

$$0. \quad \cos 7\theta - 2(25+8-30)\cos 5\theta + (20-30)\cos 3\theta$$

and if the coefficient of $\cos 3\theta$ be p_3, this gives rise to

$$p_3[(12-30)\cos 5\theta - 2(9+8-30)\cos 3\theta + (6-30)\cos\theta]$$

and if the coefficient of $\cos\theta$ be p_5, this gives rise to

$$p_5[(2-30)\cos 3\theta - 2(1+8-30)\cos\theta + (0-30)\cos\theta].$$

Hence we see that to make the coefficients vanish

of $\cos 5\theta$, we must have $-6 - 18p_3 = 0$

of $\cos 3\theta$,, $-10 + 26p_3 - 28p_5 = 0$

of $\cos$,, $-24p_3 + 42p_5 - 30p_5 = 0$

whence

$$p_3 = -\tfrac{1}{3}, \quad p_5 = -\tfrac{56}{84} = -\tfrac{2}{3}$$

and the expression

$$\cos 5\theta - \tfrac{1}{3}\cos 3\theta - \tfrac{2}{3}\cos\theta$$

only differs from the former by a numerical factor.

Erratum in Mr. Stokes's Paper, "A Critical Comparison of the Overlapping Section of the Oxford and Potsdam Astrographic Catalogues":—

Page 127, line 10,
for less *read* greater.